Derivative-Free Model Reference Adaptive Control

Tansel Yucelen* and Anthony J. Calise†
Georgia Institute of Technology, Atlanta, Georgia 30332-0150

A derivative-free, delayed weight update law is developed for model reference adaptive control of continuous-time uncertain systems, without assuming the existence of constant ideal weights. Using a Lyapunov–Krasovskii functional it is proven that the error dynamics are uniformly ultimately bounded, without the need for modification terms in the adaptive law. Estimates for the ultimate bound and the exponential rate of convergence to the ultimate bound are provided. Also discussed are employing various modification terms for further improving performance and robustness of the adaptively controlled system. Examples illustrate that the proposed derivative-free model reference adaptive control law is advantageous for applications to systems that can undergo a sudden change in dynamics.

I. Introduction

DIRECT model reference adaptive controllers require less modeling information than robust controllers and can address system uncertainties and system failures. These controllers directly adapt controller parameters in response to system variations for the purpose of canceling the effect of modeling uncertainty, without necessarily estimating the parameters of the unknown system. This property distinguishes them from adaptive controllers that employ an estimation algorithm to estimate the unknown system parameters and employ a controller that depends on the estimated parameters. This paper develops an adaptive control law that is particularly well suited for direct adaptive control in the presence of sudden or rapid time-varying changes in uncertain system dynamics.

In the past decades, numerous model reference adaptive control (MRAC) approaches have been proposed that deal with multivariable uncertain systems in continuous-time (see [1–10] and references therein). These approaches are derived based on Lyapunov theory and either assume or derive a weight update law in the form of an ordinary differential equation for the weight estimates. All these methods have in common the underlying assumption that there exists constant, but unknown, ideal set of weights. Although this assumption seems reasonable and these MRAC architectures work well on many systems, in some failure modes they may require the use of unrealistically high adaptation gain, or may fail to achieve the desired level of performance in terms of failure recovery. MRAC laws that require high gain can excite unmodeled dynamics, typically exhibit an excessive amount of control activity [11,12], amplify the effect of sensor noise, and are not sufficiently robust to time delay [13].

In this paper, we develop a derivative-free model reference adaptive control (DF-MRAC) law, which uses the information of delayed weight estimates and the information of current system states and errors. The proposed method is an extension of the iterative learning approach adopted in [14] for purposes of adaptive observer design. We relax the assumption of constant unknown ideal weights to the existence of time-varying weights, such that fast and possibly discontinuous variation in weights are allowed. The proposed

derivative-free adaptive control law is advantageous for applications to systems that can undergo a sudden change in dynamics, such as might be due to reconfiguration, deployment of a payload, docking, or structural damage. We prove that the error dynamics are uniformly ultimately bounded using a Lyapunov–Krasovskii functional, without employing modification terms in the adaptive law. We consider constant unknown ideal weights as a special case and show that the state tracking error dynamics are asymptotically stable. Finally, we discuss employing various modification terms for further improving the performance and robustness of the adaptively controlled system.

DF-MRAC differs from MRAC in that it does not make use of integration in its weight update law. In fact, DF-MRAC challenges the conventional wisdom of expecting an adaptive law to cancel the effect of matched constant disturbances and uncertainties, since one does not need adaptation to correct for biases. Biases can more reliably be handled using a nonadaptive controller with integral action. This ensures that errors in the regulated output variables go to zero for constant disturbances and constant model errors even if they are unmatched, so long as the closed loop system remains stable. Therefore, since many existing guidance and flight controllers do employ integral action, and if the goal is to augment an existing controller, then it is desirable that the weight update law not have the effect of integral action. Otherwise a conflict may arise that results in a slowly varying tracking error. In these circumstances, the role of adaptation should be confined to things like maintaining stability, error transient performance, fast upset recovery in the event there is a destabilizing failure, and preserving to the extent possible the time-delay margins of the nominal design. DF-MRAC is particularly well suited for these circumstances.

The notation used in this paper is fairly standard. $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{n \times m}$ denotes the set of $n \times m$ real matrices, $(\cdot)^T$ denotes transpose, and $(\cdot)^{-1}$ denotes inverse. Furthermore, we write $\lambda_{\min}(M)$ [respectively, $\lambda_{\max}(M)$] for the minimum (respectively, maximum) eigenvalue of M, $|\cdot|$ for the Euclidian vector norm, $\|\cdot\|$ for the Frobenius matrix norm, $\text{vec}(\cdot)$ for the column stacking operator, $\mathcal{R}\{\cdot\}$ for the range space of a matrix, $\text{diag}[A, B]$ for a block diagonal matrix formed with matrices A and B on the diagonal, and (a, b) for the open interval in $\mathbb{R}$ from a to $b > a$.

The organization of the paper is as follows. Section II provides preliminaries related to standard MRAC. Section III presents the DF-MRAC law and its stability properties for the case of systems with matched uncertainty and known control effectiveness. Section IV discusses how to employ various modifications to the DF-MRAC law. Section V presents extensions to a class of systems in which there is uncertainty in the control effectiveness. Section VI treats a first-order example to illustrate the advantages of the proposed approach in a simplified setting, and Sec. VII presents a detailed application to aircraft flight control using the generic transport model (GTM), a high-fidelity scaled transport aircraft model developed at NASA Langley Research Center [15] that models the effects of airframe damage, actuator failures, and time delay. The GTM model

*Graduate Research Assistant, School of Aerospace Engineering; tansel@gatech.edu. Student Member AIAA.
†Professor, School of Aerospace Engineering; anthony.calise@aerospace. gatech.edu. Fellow AIAA.

is representative of the NASA Airborne Subscale Trasport Aircraft Research (AirSTAR) flight-test vehicle [16,17]. DF-MRAC has been flight-tested on AirSTAR, and the results are reported in [18]. Finally, Sec. VIII summarizes the conclusions.

II. Preliminaries

In this section we state standard results for the MRAC problem. Consider the uncertain system given by

$$\dot{x}(t) = Ax(t) + B[u(t) + \Delta(x(t))] \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are known matrices, and $\Delta \colon \mathbb{R}^n \to \mathbb{R}^m$ is a matched uncertainty. Furthermore, we assume that the pair (A, B) is controllable, $x(t)$ is available for feedback, and $u(t)$ is restricted to the class of admissible controls consisting of measurable functions.

The reference model is given by

$$\dot{x}_m(t) = A_m x_m(t) + B_m r(t) \tag{2}$$

where $x_m(t) \in \mathbb{R}^n$ is the reference state vector, $r(t) \in \mathbb{R}^r$ is a bounded piecewise continuous reference input, $A_m \in \mathbb{R}^{n \times n}$ is Hurwitz, and $B_m \in \mathbb{R}^{n \times r}$ with $r \le m$. Since $r(t)$ is bounded, it follows that x_m is uniformly bounded for all $x_m(0)$.

Assumption 1. The matched uncertainty in Eq. (1) can be linearly parameterized as

$$\Delta(x) = W^T \beta(x) + \varepsilon(x), \qquad |\varepsilon(x)| \le \varepsilon^*, \qquad x \in \mathcal{D}_x \tag{3}$$

where $W \in \mathbb{R}^{s \times m}$ is the unknown *constant* weight matrix, $\beta \colon \mathbb{R}^n \to \mathbb{R}^s$ is a known vector of basis functions of the form $\beta(x) = [\beta_1(x), \beta_2(x), \ldots, \beta_s(x)]^T \in \mathbb{R}^s$, $\varepsilon \colon \mathbb{R}^n \to \mathbb{R}^m$ is the residual error, and $\mathcal{D}_x$ is a sufficiently large compact set $\mathcal{D}_x \in \mathbb{R}^n$.

Consider the following feedback control law:

$$u(t) = u_n(t) - u_{\text{ad}}(t) \tag{4}$$

where $u_n(t)$ is a nominal feedback control given by

$$u_n(t) = K_1 x(t) + K_2 r(t) \tag{5}$$

where $K_1 \in \mathbb{R}^{m \times n}$ and $K_2 \in \mathbb{R}^{m \times r}$ are nominal control gains such that $A + BK_1$ is Hurwitz, and $u_{\text{ad}}(t)$ is the adaptive feedback control component given by

$$u_{\text{ad}}(t) = \hat{W}^T(t) \beta(x(t)) \tag{6}$$

where $\hat{W}(t) \in \mathbb{R}^{s \times m}$ is an estimate of W satisfying the weight update law:

$$\dot{\hat{W}}(t) = \gamma[\beta(x(t))e^T(t)PB + \dot{\hat{W}}_m(t)], \qquad \gamma > 0 \tag{7}$$

where

$$e(t) \equiv x(t) - x_m(t) \tag{8}$$

is the state tracking error, $P \in \mathbb{R}^{n \times n}$ is the positive-definite solution of the Lyapunov equation

$$0 = A_m^T P + PA_m + Q \tag{9}$$

for any $Q = Q^T > 0$, and $\dot{\hat{W}}_m(t) \in \mathbb{R}^{s \times m}$ is a modification term, e.g.,

$$\dot{\hat{W}}_m(t) = -\sigma \hat{W}(t) \tag{10}$$

for σ modification [1], or

$$\dot{\hat{W}}_m(t) = -\sigma |e(t)| \hat{W}(t) \tag{11}$$

for e modification [2], where σ is a positive fixed gain.

Assumption 2. A_m and B_m in Eq. (2) are chosen so that

$$A_m = A + BK_1 \tag{12}$$

$$B_m = BK_2 \tag{13}$$

The dynamics in Eq. (1) can be written as

$$\dot{x}(t) = A_m x(t) + B_m r(t) + B\tilde{W}^T(t)\beta(x(t)) + B\varepsilon(x(t)) \tag{14}$$

where

$$\tilde{W}(t) \equiv W - \hat{W}(t) \tag{15}$$

is the weight update error. The state tracking error can likewise be written as

$$\dot{e}(t) = A_m e(t) + B\tilde{W}^T(t)\beta(x(t)) + B\varepsilon(x(t)) \tag{16}$$

Theorems that provide sufficient conditions under which the closed-loop system errors $e(t)$ and $\tilde{W}(t)$ are uniformly ultimately bounded (UUB) [19] for the σ and e modification cases can be found in [1,2]. The typical Lyapunov function candidate used for stability analysis of the adaptive law in Eq. (7) has the quadratic form:

$$\mathcal{V}(e, \tilde{W}) = e(t)^T P e(t) + \frac{1}{\gamma} \text{tr}[\tilde{W}(t)^T \tilde{W}(t)], \qquad \gamma > 0 \tag{17}$$

III. Derivative-Free Adaptive Control

The following assumption strengthens Assumption 1 by setting $\varepsilon(x(t)) = 0$, which can be justified under the assumption that time variation is allowed in the unknown ideal weight matrix.

Assumption 3. The matched uncertainty in Eq. (1) can be linearly parameterized as

$$\Delta(t, x(t)) = W^T(t)\beta(x(t)), \qquad x \in \mathcal{D}_x \tag{18}$$

where $W(t) \in \mathbb{R}^{s \times m}$ is an unknown *time-varying* weight matrix that satisfies $\|W(t)\| \le w^*$ and $\beta \colon \mathbb{R}^n \to \mathbb{R}^s$ is a vector of known functions of the form $\beta(x) = [\beta_1(x), \beta_2(x), \ldots, \beta_s(x)]^T \in \mathbb{R}^s$.

Remark 1. Assumption 3 expands the class of uncertainties that can be represented by a given set of basis functions. That is, an adaptive law designed subject to Assumption 3 can be more effective than an adaptive law designed subject to Assumption 1 in dominating a wider class of uncertainties, due to the fact that time variation is allowed in the unknown ideal weight matrix.

Remark 2. Assumption 3 does not place any restriction on the time derivative of the weight matrix. However, the degree of time dependence will depend on how $\beta(x)$ is chosen.

The following theorem presents the main result of this paper.

Theorem 1. Consider the uncertain system given by Eq. (1) subject to Assumption 3. Consider, in addition, the feedback control law given by Eq. (4), with the nominal feedback control component given by Eq. (5) subject to Assumption 2, and with the adaptive feedback control component given by Eq. (6), which has a derivative-free weight update law in the form

$$\hat{W}(t) = \Omega_1 \hat{W}(t - \tau) + \hat{\Omega}_2(t) \tag{19}$$

where $\tau > 0$, and $\Omega_1 \in \mathbb{R}^{s \times s}$ and $\hat{\Omega}_2 \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{s \times m}$ satisfy

$$0 \le \Omega_1^T \Omega_1 < I \tag{20}$$

$$\hat{\Omega}_2(t) = \kappa_2 \beta(x(t))e^T(t)PB, \qquad \kappa_2 > 0 \tag{21}$$

with $P \in \mathbb{R}^{n \times n}$ satisfying Eq. (9) for any symmetric $Q > 0$. Then $e(t)$ and $\tilde{W}(t)$ are UUB.

Proof. Using Eq. (19) and defining

$$\Omega_2(t) \equiv W(t) - \Omega_1 W(t - \tau) \tag{22}$$

where $\|\Omega_2(t)\| \le \delta^*$, $\delta^* = w^*(1 + \|\Omega_1(t)\|)$, and the weight update error in Eq. (15) can be rewritten as

$$\tilde{W}(t) = \Omega_1 \tilde{W}(t - \tau) + \Omega_2(t) - \hat{\Omega}_2(t) \tag{23}$$

Using Eq. (23), the state tracking error dynamics in Eq. (16) under Assumption 3 become

$$\dot{e}(t) = A_m e(t) + B[\Omega_1 \tilde{W}(t-\tau) - \hat{\Omega}_2(t) + \Omega_2(t)]^T \beta(x(t)) \quad (24)$$

To show that the closed-loop system given by Eqs. (23) and (24) is UUB, consider the Lyapunov–Krasovskii functional [20]:

$$\mathcal{V}(e(t), \tilde{W}_t) = e^T(t) P e(t) + \rho \operatorname{tr}\left[\int_{t-\tau}^t \tilde{W}^T(s) \tilde{W}(s)\, ds\right] \quad (25)$$

where $\rho > 0$ and $\tilde{W}_t$ represents $\tilde{W}(t)$ over the time interval $t - \tau$ to t. The directional derivative of Eq. (25) along the closed-loop system trajectories of Eqs. (23) and (24) is given by

$$\dot{\mathcal{V}}(e(t), \tilde{W}_t) = -e^T(t) Q e(t) + 2 e^T(t) P B [\Omega_1 \tilde{W}(t-\tau)]^T \beta(x(t))$$
$$- 2 e^T(t) P B \hat{\Omega}_2^T(t) \beta(x(t)) + 2 e^T(t) P B \Omega_2^T(t) \beta(x(t))$$
$$+ \rho \operatorname{tr}[-\xi \tilde{W}^T(t) \tilde{W}(t) + \eta \tilde{W}^T(t) \tilde{W}(t) - \tilde{W}^T(t-\tau) \tilde{W}(t-\tau)] \quad (26)$$

where $\eta = 1 + \xi$. In what follows we impose the restriction $\xi \geq 0$.

Using Eq. (23) to expand the term $\operatorname{tr}[\eta \tilde{W}^T(t) \tilde{W}(t)]$ in Eq. (26) produces

$$\dot{\mathcal{V}}(e(t), \tilde{W}_t) = -e^T(t) Q e(t) + 2 e^T(t) P B [\Omega_1 \tilde{W}(t-\tau)]^T \beta(x(t))$$
$$- 2 e^T(t) P B \hat{\Omega}_2^T(t) \beta(x(t)) + 2 e^T(t) P B \Omega_2^T(t) \beta(x(t))$$
$$+ \rho \operatorname{tr}[-\xi \tilde{W}^T(t) \tilde{W}(t) - \tilde{W}^T(t-\tau) \tilde{W}(t-\tau)$$
$$+ \eta \tilde{W}^T(t-\tau) \Omega_1^T \Omega_1 \tilde{W}(t-\tau) + \eta \hat{\Omega}_2^T(t) \hat{\Omega}_2(t)$$
$$+ \eta \Omega_2^T(t) \Omega_2(t) - 2\eta \hat{\Omega}_2^T(t) \Omega_1 \tilde{W}(t-\tau)$$
$$+ 2\eta \tilde{W}^T(t-\tau) \Omega_1^T \Omega_2(t) - 2\eta \hat{\Omega}_2^T(t) \Omega_2(t)] \quad (27)$$

Next, consider the fact $a^T b \leq \gamma a^T a + b^T b / 4\gamma$ ($\gamma > 0$), which follows from Young's inequality [14,21] for any vectors a and b. This can be generalized to matrices as

$$\operatorname{tr}[A^T B] = \operatorname{vec}(A)^T \operatorname{vec}(B) \leq \gamma \operatorname{vec}(A)^T \operatorname{vec}(A)$$
$$+ \operatorname{vec}(B)^T \operatorname{vec}(B) / 4\gamma = \gamma \operatorname{tr}[A^T A] + \operatorname{tr}[B^T B] / 4\gamma$$

($\gamma > 0$) for any matrices A and B with appropriate dimensions. Using this, we can write

$$\operatorname{tr}[2\eta \tilde{W}^T(t-\tau) \Omega_1^T \Omega_2(t)] \leq \operatorname{tr}[\gamma \tilde{W}^T(t-\tau) \Omega_1^T \Omega_1 \tilde{W}(t-\tau)]$$
$$+ \operatorname{tr}[\eta^2 \Omega_2^T(t) \Omega_2(t) / \gamma], \qquad \gamma > 0 \quad (28)$$

Using Eq. (21) with $\kappa_2 \equiv 1/\rho\eta > 0$ for $\hat{\Omega}_2(t)$, and substituting Eq. (28) in Eq. (27), it follows that

$$\dot{\mathcal{V}}(e(t), \tilde{W}_t) \leq -e^T(t) Q e(t) - \kappa_2 e^T(t) P B B^T P e(t) \beta(x(t))^T \beta(x(t))$$
$$- \rho \xi \operatorname{tr}[\tilde{W}^T(t) \tilde{W}(t)] - \rho \operatorname{tr}[\tilde{W}^T(t-\tau)[I - (\eta + \gamma)\Omega_1^T \Omega_1]$$
$$\times \tilde{W}(t-\tau)] + \rho(\eta + \eta^2/\gamma)\operatorname{tr}[\Omega_2^T(t)\Omega_2(t)] \quad (29)$$

Using Eq. (20) with $\kappa_1 \equiv 1/(\eta + \gamma) < 1$ for Ω_1 yields

$$\dot{\mathcal{V}}(e(t), \tilde{W}_t) \leq -c_1 |e(t)|^2 - c_2 \|\tilde{W}(t)\|^2 - c_3 \|\tilde{W}(t-\tau)\|^2 + d \quad (30)$$

where the constants c_1, c_2, c_3, and d are given by

$$c_1 = \lambda_{\min}(Q) > 0 \quad (31)$$

$$c_2 = \rho\xi \geq 0 \quad (32)$$

$$c_3 = \rho\lambda_{\min}(I - \kappa_1^{-1}\Omega_1^T \Omega_1) > 0 \quad (33)$$

$$d = \rho(\eta + \eta^2/\gamma)\delta^{*2} \geq 0 \quad (34)$$

where $\rho = 1/\kappa_2\eta$. If $\xi > 0$, then since $\eta = \kappa_1^{-1} - \gamma = 1 + \xi$, $0 < \kappa_1 < 1$, and $\gamma > 0$, it follows that η must lie in the open interval $(1, 1/\kappa_1)$. Either $|e(t)| > \Psi_1$ or $\|\tilde{W}(t)\| > \Psi_2$ or $\|\tilde{W}(t-\tau)\| > \Psi_3$ renders $\dot{\mathcal{V}}(e(t), \tilde{W}_t) < 0$, where $\Psi_1 = \sqrt{d/c_1}$, $\Psi_2 = \sqrt{d/c_2}$, and $\Psi_3 = \sqrt{d/c_3}$ or, equivalently,

$$\Psi_1 = \delta^* \sqrt{\rho(\eta + \eta^2/\gamma)/\lambda_{\min}(Q)} = \delta^* / \sqrt{\kappa_2\lambda_{\min}(Q)[1 - \eta\kappa_1]} \quad (35)$$

$$\Psi_2 = \Psi_1 \sqrt{c_1/c_2} = \Psi_1 \sqrt{\kappa_2\lambda_{\min}(Q)\eta/[\eta - 1]} \quad (36)$$

$$\Psi_3 = \Psi_1 \sqrt{c_1/c_3} = \Psi_1 \sqrt{\kappa_1\kappa_2\lambda_{\min}(Q)\eta/\lambda_{\min}(\kappa_1 I - \Omega_1^T \Omega_1)} \quad (37)$$

Hence, it follows that $e(t)$ and $\tilde{W}(t)$ are UUB. The proposed adaptive control architecture is shown in Fig. 1.

Remark 3. Letting $\dot{\hat{W}}_m(t) = 0$ and using a first-order Euler method for integration in Eq. (7), with τ_s being the step size, results in

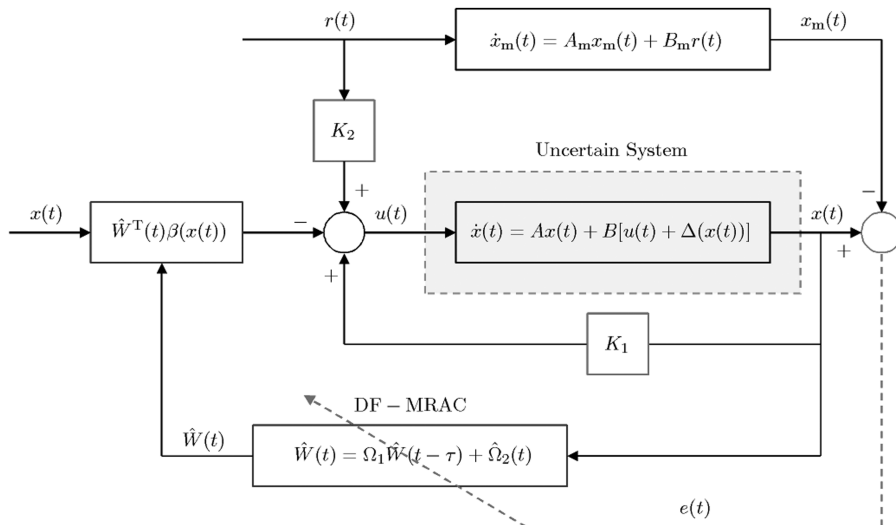$$\hat{W}(t) = \gamma\tau_s(\beta(x(t))e^T(t) P B) + \hat{W}(t - \tau_s) \quad (38)$$

Fig. 1   Visualization of the proposed DF-MRAC architecture.

This form of weight update law is identical to the DF-MRAC law in Eq. (19), if $\Omega_1 = I$, $\kappa_2 = \gamma\tau_s$, and $\tau = \tau_s$, with the exception that the choice $\Omega_1 = I$ is not permitted in DF-MRAC. In DF-MRAC, Ω_1 can be chosen, for example, as ςI, where $0 < |\varsigma| < 1$, and τ is not necessarily equal to τ_s. This added dimension in the tuning process provides memory to the adaptive law.

Remark 4. The derivative-free weight update law given by Eq. (19) subject to Eqs. (20) and (21) does not require a modification term to prove the error dynamics, including the weight errors, are UUB.

Define $q(t) \equiv [e^T(t), \tilde{v}(t, \tau)]^T$, where

$$\tilde{v}^2(t, \tau) \equiv \text{tr}\left[\int_{-\tau}^0 \tilde{W}^T(s)\tilde{W}(s)\,ds\right]$$

and let $\mathcal{B}_r = \{q(t): \|q(t)\| < r\}$, such that $\mathcal{B}_r \subset \mathcal{D}_q$ for a sufficiently large compact set $\mathcal{D}_q$. Then we have the following corollary.

Corollary 1. Under the conditions of Theorem 1, an estimate for the ultimate bound, for the case $\xi > 0$, is given by

$$r = \sqrt{\frac{\lambda_{\max}(P)\Psi_1^2 + \rho\tau\Psi_2^2}{\lambda_{\min}(\tilde{P})}} \tag{39}$$

where $\tilde{P} = \text{diag}[P, \rho]$.

Proof. Denote $\Omega_\alpha = \{q(t) \in \mathcal{B}_r : q^T(t)\tilde{P}q(t) \le \alpha\}$, $\alpha = \min_{\|q(t)\|=r} q^T(t)\tilde{P}q(t) = r^2\lambda_{\min}(\tilde{P})$. Since

$$\mathcal{V}(e(t), \tilde{W}_t) = q^T(t)\tilde{P}q(t) = e^T(t)Pe(t)$$
$$+ \rho\,\text{tr}\left[\int_{t-\tau}^t \tilde{W}^T(s)\tilde{W}(s)\,ds\right] \tag{40}$$

it follows that Ω_α is an invariant set as long as

$$\alpha \ge \lambda_{\max}(P)\Psi_1^2 + \rho\tau\Psi_2^2 \tag{41}$$

Thus, the minimum size of $\mathcal{B}_r$ that ensures this condition has radius given by Eq. (39). The sets used in this proof are illustrated in Fig. 2.

Remark 5. The proofs of Theorem 1 and Corollary 1 assume that the sets $\mathcal{D}_x$ and $\mathcal{D}_q$ are sufficiently large. If we define $\mathcal{B}_{r^*}$ as the largest ball contained in $\mathcal{D}_q$, and assume that the initial conditions are such that $q(0) \subset \mathcal{B}_{r^*}$, then from Fig. 2 we have the added condition that $r < r^*$, which implies a lower bound on ρ. It can be shown that in this case the lower bound must be such that $\lambda_{\min}(\tilde{P}) = \rho$. With r defined by Eq. (39) and $\lambda_{\min}(\tilde{P}) = \rho$, the condition $r < r^*$ implies

$$\rho > \frac{\bar{\lambda}(P)\Psi_1^2}{r^{*2} - \tau\Psi_2^2} \tag{42}$$

Since $\kappa_2 = 1/\rho\eta$ and $\eta > 1$, it follows from Eq. (42) that r^* should ensure that

$$\kappa_2 < \frac{r^{*2} - \tau\Psi_2^2}{\bar{\lambda}(P)\Psi_1^2} \tag{43}$$

Therefore, the meaning of $\mathcal{D}_q$ *sufficiently large* in Corollary 1 is that
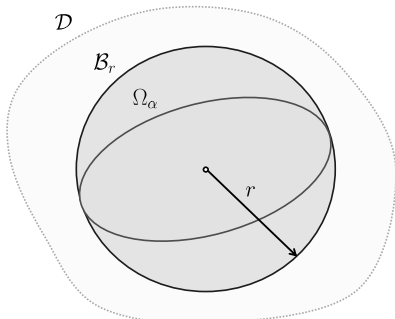

Fig. 2   Geometric representation of sets.

$$r^* > \sqrt{\kappa_2\lambda_{\max}(P)\Psi_1^2 + \tau\Psi_2^2}$$

and $q(0) \subset \mathcal{D}_{r^*}$. The meaning of $\mathcal{D}_x$ sufficiently large is difficult to characterize precisely, since $x(t)$ depends on both $r(t)$ and $x(0)$. Nevertheless it can be seen that increasing κ_2 implies increasing the require size of the set $\mathcal{D}_x$.

Corollary 2. Under the conditions of Theorem 1, the error trajectory approaches the ultimate bound exponentially in time according to

$$|q(t)| \le k|q(0)|e^{-c_r t}, \qquad t < T \tag{44}$$

with a convergence rate given by

$$c_r = \frac{\tau/(1+\tau)\Psi_1^2}{2\lambda_{\max}(\tilde{P})}, \qquad \tau = \frac{\rho(\eta-1)}{\lambda_{\min}(Q)} \tag{45}$$

Proof. Choosing ξ so that $c_2 = c_1\tau$ and using the expressions for c_1 and c_2 in Eqs. (31) and (32) results in the expression for τ in Eq. (45). Then from Eq. (30), we can write

$$\dot{\mathcal{V}}(e(t), \tilde{W}_t) \le -c_1|q(t)|^2 + d \tag{46}$$

Define $\hat{c} \equiv d/\mu^2$, where $\mu \equiv \sqrt{\Psi_1^2 + \tau\Psi_2^2}$. Then when $|q(t)| > \mu$, we have that

$$\dot{\mathcal{V}}(e(t), \tilde{W}_t) \le -(c_1 - \hat{c})|q(t)|^2 \le -k_3|q(t)|^2 \tag{47}$$

where $k_3 \equiv c_1 - \hat{c}$. Now, $c_1 = d/\Psi_1^2$ and $c_2 = d/\Psi_2^2$, and since $c_2/c_1 = \tau$, it follows that $\Psi_1^2 = \tau\Psi_2^2$, and therefore $\hat{c} = d/(1+\tau)\Psi_1^2$. Therefore, $k_3 = \tau/(1+\tau)\Psi_1^2$. Finally, since $k_1|q(t)|^2 \le \mathcal{V}(e(t), \tilde{W}_t) \le k_2|q(t)|^2$, where $k_1 = \lambda_{\min}(\tilde{P}) > 0$, $k_2 = \lambda_{\max}(\tilde{P}) > 0$, and $\dot{\mathcal{V}}(e(t), \tilde{W}_t) \le -k_3|q(t)|^2$, then Eq. (44) follows directly from Corollary 5.3 of [22], where

$$k \equiv \sqrt{k_2/k_1} = \sqrt{\lambda_{\max}(\tilde{P})/\lambda_{\min}(\tilde{P})}$$

and $c_r = (k_3/2k_2) = \tau/2(1+\tau)\lambda_{\max}(\tilde{P})\Psi_1^2$. □

For the case of constant ideal weights in Assumption 3, Theorem 1 specializes to the following theorem. In this case, we assume that the uncertainty is structured. That is, the vector of known functions in Eq. (18) represent the vector of *known* basis functions.

Theorem 2. Consider the uncertain system given by Eq. (1) subject to Assumption 3, where $W \in \mathbb{R}^{s \times m}$ is an unknown *constant* weight matrix. Consider, in addition, the feedback control law given by Eq. (4), with the nominal feedback control component given by Eq. (5) subject to Assumption 2, and with the adaptive feedback control component given by Eq. (6), which has the derivative-free weight update law in the form of Eqs. (19) and (21), where $\Omega_1 = I$. Then $e(t)$ and $\tilde{W}(t)$ approach a subspace in these error variables in which $e(t) = 0$ and $B\tilde{W}^T(t)\beta(x(t)) = 0$.

Proof. The result follows directly from the proof of Theorem 1 by choosing $\Omega_1 = I$. In this case, $\delta^* = 0$ due to $\Omega_2(t) = 0$ in Eq. (22), which follows from the fact that the ideal weights are constant, and $\Omega_1 = I$. Then the inequality in (28) is not needed, since the left-hand side vanishes. In this case, $\kappa_1 = 1/\eta = 1$, $\xi = 0$, $c_2 = 0$ in Eq. (32), $c_3 = 0$ in Eq. (33), and $d = 0$ in Eq. (34). Therefore, it follows from Eq. (30) that the entire error space is invariant. Let $\mathcal{E}$ denote the set of points in this space where $\dot{\mathcal{V}}(e(t), \tilde{W}_t) = 0$. From Eq. (30) all points in lie in a subspace, where $e(t) = 0$ and it follows from LaSalle's theorem [22] that all solutions in the error space approach the largest invariant set $\mathcal{M}$ in $\mathcal{E}$. Now $e(t) = 0$ implies that $\hat{\Omega}_2(t) = 0$ and $\dot{e}(t) = 0$. Then Eq. (24), with all of the above taken together, implies that $\mathcal{M}$ is composed of all points in the error space in which $e(t) = 0$ and $B\tilde{W}^T(t)\beta(x(t)) = 0$. □

Remark 6. The system is said to be sufficiently excited if $r(t)$ is such that the conditions $e(t) = 0$, $B\tilde{W}^T(t)\beta(x(t)) = 0$ admit only the solution $\tilde{W}(t) = 0$ in the limit $t \to \infty$. It is straightforward to show that this amounts to the standard MRAC condition for persistency of excitation [23].

Remark 7. As noted in the Introduction, DF-MRAC does not employ an integrator in its weight update law. This is advantageous from the perspective of augmenting a nominal controller that employs integral action to ensure that the regulated output variables track $r(t)$ for constant disturbances, regardless of how these disturbances may enter the system. An example that illustrates this advantage is provided in Sec. V.

Remark 8. The expressions for Ψ_1 and Ψ_2 represent ultimate bounds for $|e(t)|$ and $\|\tilde{W}(t)\|$, respectively. These expressions depend on $\eta > 1$. In [24] it is shown that there exists an optimal value η^* such that the convergence rate c_r in Eq. (45) attains a maximum on the open interval $\{1, 1/\kappa_1\}$, and curves are provided that show the tradeoff between the UUB and c_r for a range of values for κ_1 and κ_2.

IV. Modifications to Derivative-Free Adaptive Control

Although the derivative-free weight update law does not require a modification term to prove the error dynamics are UUB, one may wish to employ a modification term in order to improve performance or robustness of the system. The following theorem extends Theorem 1 to a general form of modified DF-MRAC.

Theorem 3. Consider the uncertain system given by Eq. (1) subject to Assumption 3. Consider, in addition, the feedback control law given by Eq. (4), with the nominal feedback control component given by Eq. (5) subject to Assumption 2, and with the adaptive feedback control component given by Eq. (6), which has a derivative-free weight update law in the form given by Eq. (19), where $\tau > 0$ is a time-delay design value, $\Omega_1 \in \mathbb{R}^{s \times s}$, and $\hat{\Omega}_2(t) \equiv \hat{\Omega}_2(x(t), e(t))$ and $\hat{\Omega}_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{s \times m}$, satisfy

$$0 < \Omega_1^T \Omega_1 < \kappa_1 I, \qquad 0 < \kappa_1 < 1 \qquad (48)$$

$$\hat{\Omega}_2(t) = \kappa_2[\beta(x(t))e^T(t)PB - \kappa_m S(t)\hat{W}(t)], \qquad \kappa_m > 0 \qquad (49)$$

where κ_m is the modification gain, $P \in \mathbb{R}^{n \times n}$ satisfies

$$0 = A_m^T P + P A_m - \kappa_m^2 P B B^T P + Q \qquad (50)$$

for any symmetric $Q > 0$, and $S(t) \in \mathbb{R}^{s \times s}$ satisfies $\|S(t)\| < s^*$. Then $e(t) = x(t) - x_m(t)$ and $\tilde{W}(t) = W(t) - \hat{W}(t)$ are UUB.

Proof. The result follows directly from the proof of Theorem 1, with Eqs. (31–34) changed to

$$c_1 = \lambda_{\min}(Q) > 0 \qquad (51)$$

$$c_2 = \rho \xi + s^{*2} \kappa^* > 0 \qquad (52)$$

$$c_3 = \rho \lambda_{\min}(I - [1 + \eta + \gamma]\Omega_1^T \Omega_1) > 0 \qquad (53)$$

$$d = \rho(1 + \eta + \eta^2/\gamma)\delta^{*2} + \omega^{*2} s^{*2} \kappa^* > 0 \qquad (54)$$

where $\kappa^* = 1 + 2\kappa_m + (1 + \xi)^{-1}\kappa_m^2 > 0$. $\square$

Remark 9. If $S(t)$ is not assumed to be bounded by s^*, e.g., $S(t) = \|e(t)\|$ in the case of e modification [2], then one can show that $e(t)$ and $\tilde{W}(t)$ are UUB by applying a projection operator [4] to the weight estimates given by Eq. (19).

Remark 10. In the case of σ modification [1], we have $S(t) = I$ in Eq. (49). Recently an adaptive loop recovery (ALR) modification term [25] was proposed with the objective of recovering the loop transfer properties of a chosen reference system. ALR modification can be introduced by letting $S(t) = \beta_x(x(t))\beta_x^T(x(t))$, where $\beta_x(x(t)) \equiv d\beta(x(t))/dx(t) \in \mathbb{R}^{s \times n}$ and in this case $s^* = 1$. Table 1 summarizes DF-MRAC laws for this and other modification terms as well.

V. Extensions to the Input Uncertainty Case

Consider the case of input uncertainty in which the range space of B is preserved by writing the uncertain system in the form

$$\dot{x}(t) = Ax(t) + B\Lambda[u(t) + \Delta(x(t))] \qquad (55)$$

where $\Lambda \in \mathbb{R}^{m \times m}$ is an unknown diagonal matrix with diagonal elements $\lambda_i > 0$ $(i = 1, \ldots, m)$. Consider, in addition, the feedback control law given by Eq. (4), with the nominal feedback control component given by Eq. (5) subject to Assumption 2, and with the adaptive feedback control component given by

$$u_{\mathrm{ad}}(t) = -\hat{W}^T(t)\beta(x(t)) - \hat{D}(t)u_n(t) \qquad (56)$$

where $\hat{D}(t) = \mathrm{diag}[\hat{d}_1(t), \ldots, \hat{d}_m(t)]$. In this case, the state tracking error dynamics in Eq. (16) under Assumption 3 become

$$\dot{e}(t) = A_m e(t) + B\Lambda(\tilde{W}^T(t)\beta(x(t)) + \tilde{D}(t)u_n(t)) \qquad (57)$$

where $\tilde{D} \equiv D - \hat{D}$, and D is a diagonal matrix with diagonal elements $d_i = (\lambda_i - 1)/\lambda_i$ $(i = 1, \ldots, m)$. The following theorem states an extension to Theorem 1 for the case of input uncertainty.

Theorem 4. Consider the uncertain system given by Eq. (55) subject to Assumption 3. Consider, in addition, the feedback control law given by Eq. (4), with the nominal feedback control component given by Eq. (5) subject to Assumption 2, and with the adaptive feedback control component given by Eq. (56), which has derivative-free weight update laws in the form given by Eqs. (19–21), and

$$\hat{d}_i(t) = \varphi_{d_i}\hat{d}_i(t - \tau_{d_i}) + \hat{\Omega}_{d_i}(t) \qquad (58)$$

where $\tau_{d_i} > 0$ $(i = 1, \ldots, m)$, $0 \le \varphi_{d_i}^2 < 1$, and $\hat{\Omega}_{d_i} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfy

$$\hat{\Omega}_{d_i}(t) = \gamma_{d_i}u_{n_i}(t)e^T(t)PB_i, \qquad \gamma_{d_i} > 0 \qquad (59)$$

with B_i denoting the ith column of B and $P \in \mathbb{R}^{n \times n}$ satisfying Eq. (9) for any symmetric $Q > 0$. Then $e(t)$, $\tilde{W}(t)$, and $\tilde{D}(t)$ are UUB.

Proof. The result follows similar to the proof of Theorem 1 by considering the Lyapunov–Krasovskii functional:

Table 1 DF-MRAC laws for various modification terms

Modification	DF-MRAC law for $0 \le \Omega_1^T \Omega_1 < I$, $\kappa_2 > 0$, $\kappa_m > 0$
Original Eq. (19)	$\hat{W}(t) = \Omega_1 \hat{W}(t - \tau) + \kappa_2[\beta(x(t))e^T PB]$.
σ [1]	$\hat{W}(t) = \Omega_1 \hat{W}(t - \tau) + \kappa_2[\beta(x(t))e^T PB - \kappa_m \hat{W}(t)]$.
e [2]	$\hat{W}(t) = \Omega_1 \hat{W}(t - \tau) + \kappa_2[\beta(x(t))e^T PB - \kappa_m\|e(t)\|\hat{W}(t)]$.
ALR [25]	$\hat{W}(t) = \Omega_1 \hat{W}(t - \tau) + \kappa_2[\beta(x(t))e^T PB - \kappa_m \beta_x(x(t))\beta_x^T(x(t))\hat{W}(t)]$.
$\mathcal{Q}$ [26,27]	$\hat{W}(t) = \Omega_1 \hat{W}(t - \tau) + \kappa_2[\beta(x(t))e^T PB - \kappa_m(\hat{W}^T q(t) - c(t))q(t)]$; where $q(t) \equiv \int_{t_d}^t \beta(x(s))\,\mathrm{d}s$, $t_d > 0$, $c(t) \equiv e_n(t) - e_n(t_d) - \int_{t_d}^t B^T A_m \times A_m e(s)\,\mathrm{d}s + \int_{t_d}^t u_{\mathrm{ad}}(s)\,\mathrm{d}s + \int_{t_d}^t u_{\mathrm{ad}}(s)\,\mathrm{d}s$, and $e_n(t)$ is the nth component of the vector $e(t)$, and $t_d > 0$.
Optimal [28]	$\hat{W}(t) = \Omega_1 \hat{W}(t - \tau) + \kappa_2[\beta(x(t))e^T PB + \kappa_m \beta(x(t))\beta(x(t))^T \hat{W}(t)B^T P A_m^{-1}B]$.
Combined/composite model reference adaptive control [29]	$\hat{W}(t) = \Omega_1 \hat{W}(t - \tau) + \kappa_2[\beta(x(t))e^T PB - \kappa_m \beta_f(x(t))e_Y^T(t)]$; where $\beta_f(x(t)) = \mathcal{L}^{-1}\{\frac{\lambda_f}{s+\lambda_f}\}\beta(x(t))$, where $\mathcal{L}^{-1}$ denotes the Laplace operator, and $e_Y(t)$ is the predictor estimation error defined in [29].
$\mathcal{K}$ [30]	$\hat{W}(t) = \Omega_1 \hat{W}(t - \tau) + \kappa_2[\beta(x(t))e^T PB - \kappa_m \int_{t-t_d}^t \hat{W}(s)\,\mathrm{d}s]$; where $t_d > 0$

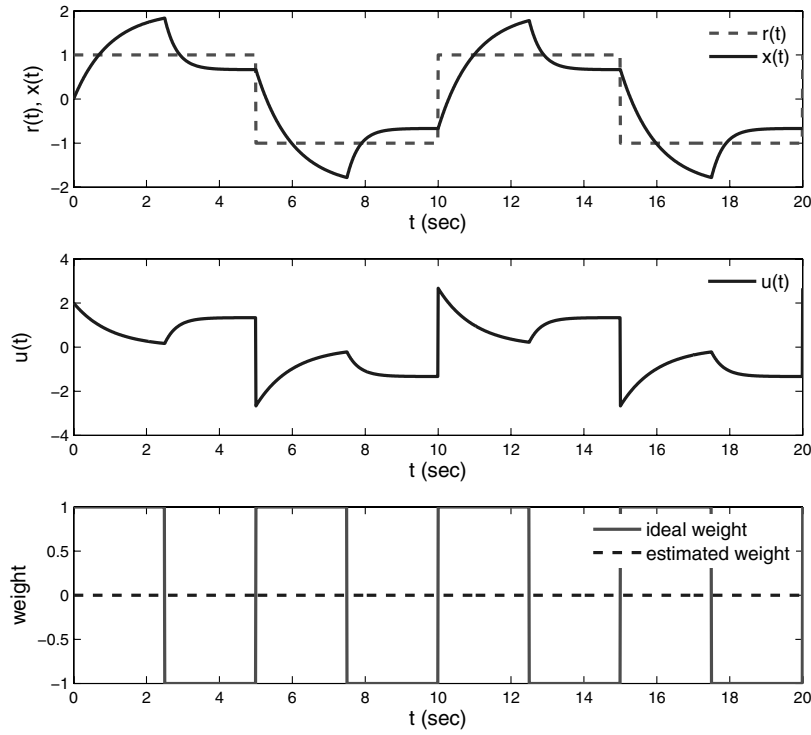**Fig. 3   Responses with nominal controller for the square wave ideal weight.**

$$\mathcal{V}(e(t), \tilde{W}_t, \tilde{D}_t) = e^T(t)Pe(t) + \rho \operatorname{tr}\left[\int_{t-\tau}^{t} \tilde{W}^T(s)\tilde{W}(s)\Lambda \, ds\right]$$
$$+ \sum_{i=1}^{m} \rho_{d_i} \operatorname{tr}\left[\int_{t-\tau_{d_i}}^{t} \lambda_i \tilde{d}_i^2(s) \, ds\right] \tag{60}$$

where $\rho_{d_i} > 0$ and $\tilde{d}_i(t) \equiv d_i - \hat{d}_i(t)$. $\square$

Remark 11. Under the conditions of Theorem 4, an estimate for the ultimate bound and convergence rate to this ultimate bound can be expressed in a form similar to Corollaries 1 and 2, respectively. For the case of constant ideal weights in Assumption 3, Theorem 4 with

$\varphi_{d_i} = 1$ $(i = 1, \ldots, m)$ guarantees that $e(t)$, $\tilde{W}(t)$, and $\tilde{D}(t)$ approach a subspace in these error variables, where $e(t) = 0$, $B\Lambda\tilde{W}^T(t)\beta(x(t)) = 0$, and $B\Lambda\tilde{D}u_n = 0$, similar to the result given by Theorem 2. Modifications to the derivative-free weight update laws in Eqs. (19) and (58) can be employed, analogous to those given in Table 1.

Remark 12. In the case where $m > r$, control allocation can be used to permit some of the diagonal elements of Λ to be zero as long as $\mathcal{R}\{B\} = \mathcal{R}\{B\Lambda\}$ [31]. The manner in which this extension can be implemented is illustrated in Sec. VII.
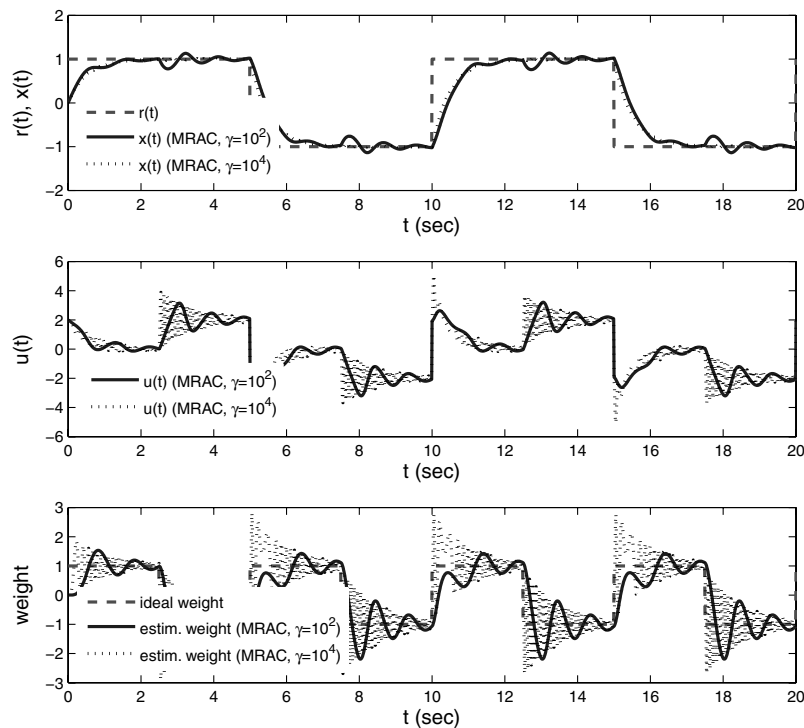

Fig. 4   Responses with standard MRAC using $\gamma = 10^2$ and 10^4.
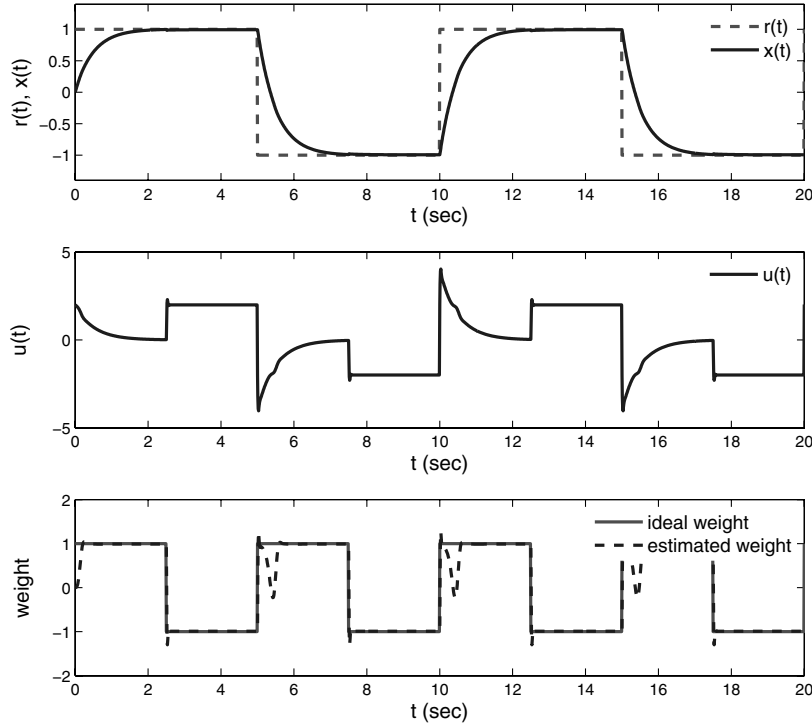
Fig. 5   Responses with DF-MRAC for the square wave ideal weight.

VI.   First-Order System Example

In this section we compare the standard MRAC law given by Eq. (7) with the proposed DF-MRAC law given by Eq. (19) on a simple model for the rolling dynamics of an aircraft [32].

A.   Uncertainty with Time-Varying Ideal Weights

Consider the scalar dynamics

$$\dot{x}(t) = L_p x(t) + L_\delta[u(t) + \Delta(x(t))] \qquad (61)$$

where $x(t)$ represents roll rate, $u(t)$ represents aileron deflection, $L_p = -1, L_\delta = 1$, and $\Delta(x(t)) = w(t)x(t)$, with $w(t)$ being a square wave having an amplitude of 1.0 and a period of 5.0 s. Although aircraft dynamics do not behave in this manner, their stability derivatives can undergo a sudden change in the event of damage to the airframe. So this example should be regarded in this context. In both the MRAC and DF-MRAC architectures, we choose as a reference model $A_m = -2$ and $B_m = 2$. Note that for this example $\varepsilon(x(t)) = 0$, so we may use the MRAC law given by Eq. (7) without a modification term ($\sigma = 0$) and choose $\beta(x(t)) = x(t)$. Furthermore, we consider two adaptation gains, $\gamma = 10^2$ and 10^4. For the
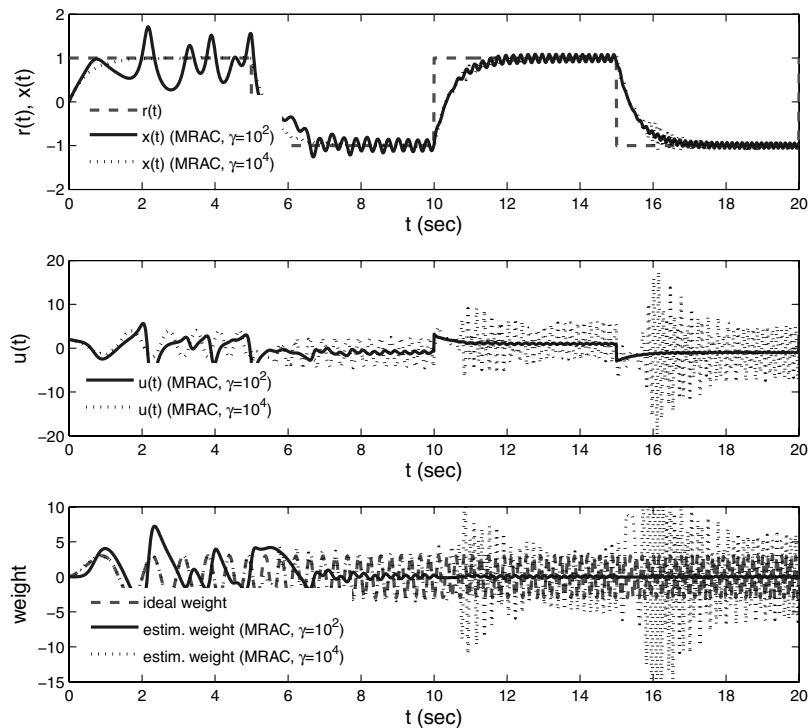

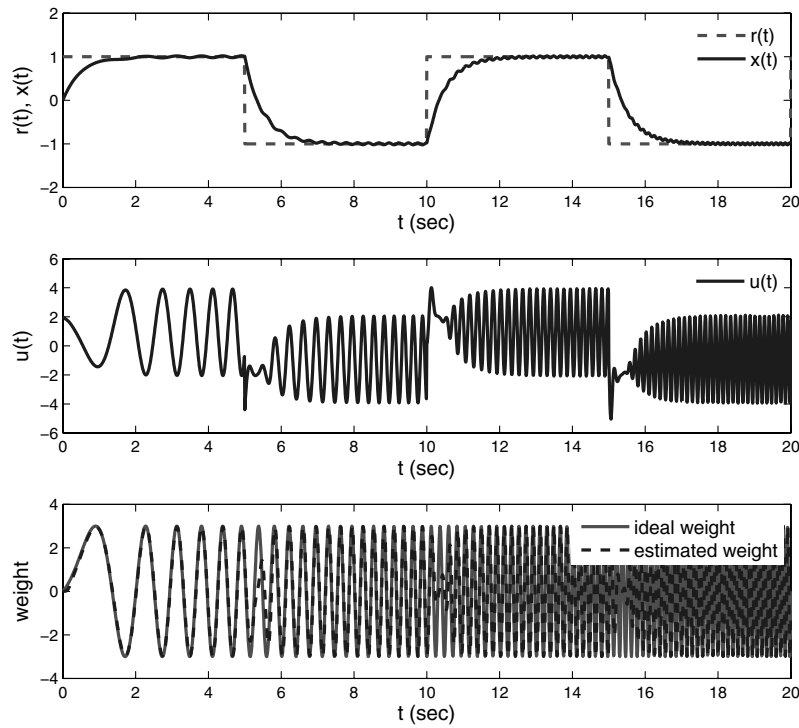Fig. 6   Responses with standard MRAC using $\gamma = 10^2$ and 10^4.

Fig. 7   Responses with DF-MRAC for a sinusoidal ideal weight.

DF-MRAC law given by Eq. (19), we set $\tau = 0.01$ s and $\Omega_1 = 0.5$, which satisfies Eq. (20), and $\kappa_2 = 200$ in Eq. (21). $Q = 2$ was used in Eq. (9) to solve for P in both architectures. Figures 3–10 present the results.

Figure 3 shows the performance of the nominal control design without adaptation. In Fig. 4, the standard MRAC architecture is used with $\gamma = 10^2$ and 10^4, respectively. Tracking performance is not improved by increasing the adaptation gain beyond 10^2, and increasing gain causes high-frequency oscillations in the control response that would be unacceptable in a real system. Figure 5 shows the case in which the DF-MRAC adaptive law in Eq. (19) was used

with $\kappa_2 = 200$. It can be seen that tracking performance is excellent, the control time history is reasonable, and the estimated weight is close to the ideal value in this case.

As a variation of the previous example, the ideal weight history was changed to a sinusoidal function in which the frequency was varied from 0.5 up to 50 rad/s. The results are shown in Figs. 6–8. In Fig. 6, note that the adaptive controller does not significantly improve the response at the low setting for adaptation gain and gives an even worse response when using the high setting for adaptation gain. Figures 7 and 8 show that the DF-MRAC case gives an excellent response, and the estimated weight is very close to the true weight,
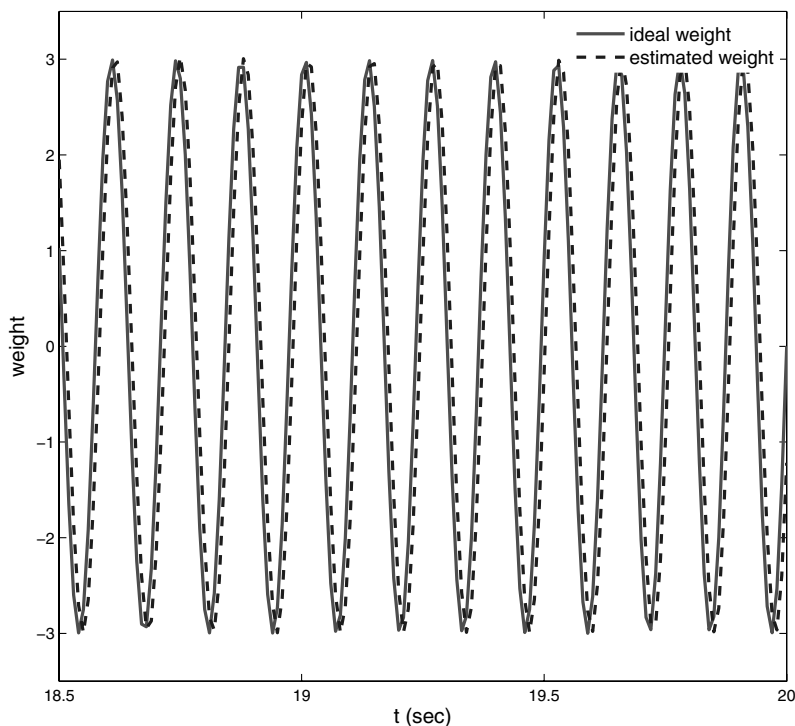

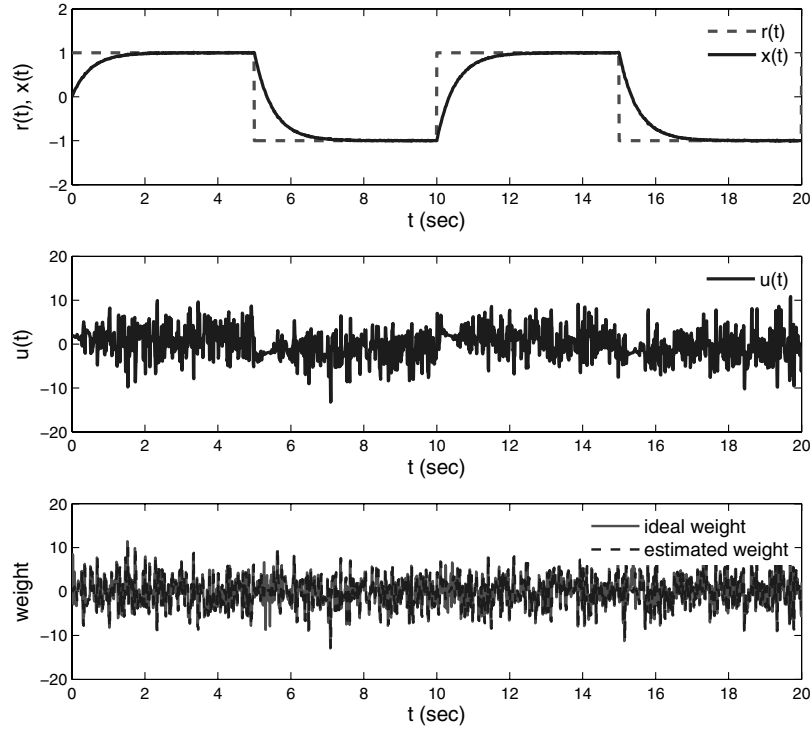Fig. 8   Expanded view of the estimate of the ideal weight.

Fig. 9 Responses with DF-MRAC for a band-limited white noise ideal weight.

even at the high-frequency end (see Fig. 8). Inspired by this result we decided to try a case in which $w(t)$ is a band-limited white noise signal. The associated results are shown in Figs. 9 and 10. In particular, Fig. 10 shows that the estimated weight is remarkably close to $w(t)$.

B. Uncertainty with Constant Ideal Weights

Consider the scalar dynamics given by Eq. (61) with $\Delta(x(t)) = 1 + 5x(t)$. To illustrate the point that conventional MRAC is problematic when augmenting a nominal controller that has integral action, we introduce an integrator state

$$\dot{x}_i(t) = -x(t) + r(t) \tag{62}$$

and consider the augmented dynamics

$$\dot{x}_a(t) = \begin{bmatrix} L_p & 0 \\ -1 & 0 \end{bmatrix} x_a(t) + \begin{bmatrix} L_\delta \\ 0 \end{bmatrix} (u(t) + \Delta(x)) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t) \tag{63}$$

where $x_a^T(t) = [\, x(t) \quad x_i(t) \,]$, and choose a proportional–integral (PI) form for the nominal controller:
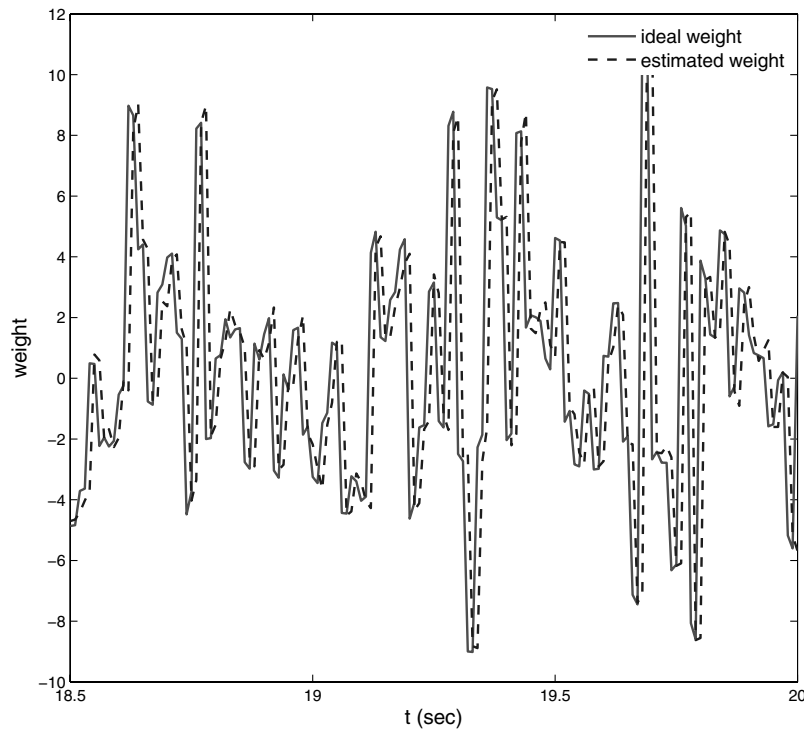

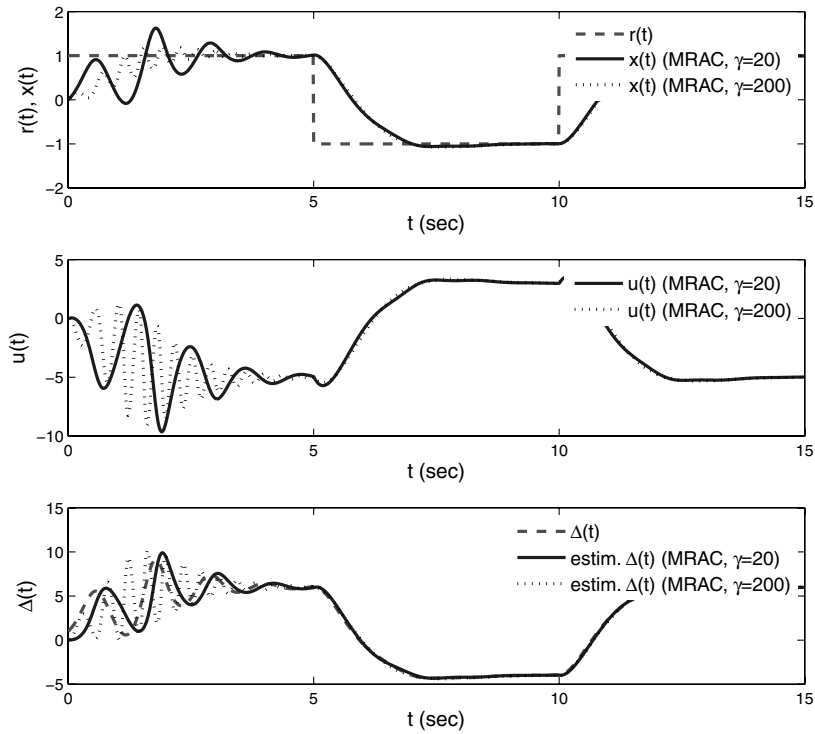
Fig. 10 Expanded view of the estimate of the ideal weight.

Fig. 11   Responses with standard MRAC for an uncertainty with constant ideal weights.

$$u_n(t) = -[\, k_x \quad k_i \,]\, x_a(t) \tag{64}$$

Then the reference model in Eq. (2) becomes

$$\dot{x}_m(t) = \begin{bmatrix} L_p - L_\delta k_x & -L_\delta k_i \\ -1 & 0 \end{bmatrix} x_m(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t) \tag{65}$$

The adaptive control is chosen in the form of Eq. (6), with $\beta^T(x(t)) = [\, 1 \quad x_a^T(t)\,]$, and uses the same weight update law as in

the previous example. We again let $L_p = -1$ and $L_\delta = 1$, and the PI gains are set to $k_x = 1.5$ and $k_i = 3$. For the MRAC law given by Eq. (7), we consider two adaptation gains, $\gamma = 20$ and 200. For the DF-MRAC law, the parameter settings are the same as in the previous example. Figures 11 and 12 present results that support the statement made in Remark 7. Figure 11 illustrates the conflict that can arise when a nominal control law with integral action is augmented with an adaptive controller that employs an integrator in its weight update law. Note the slowly varying tracking error during the first transient phase. In more complex problems this slow variation can be
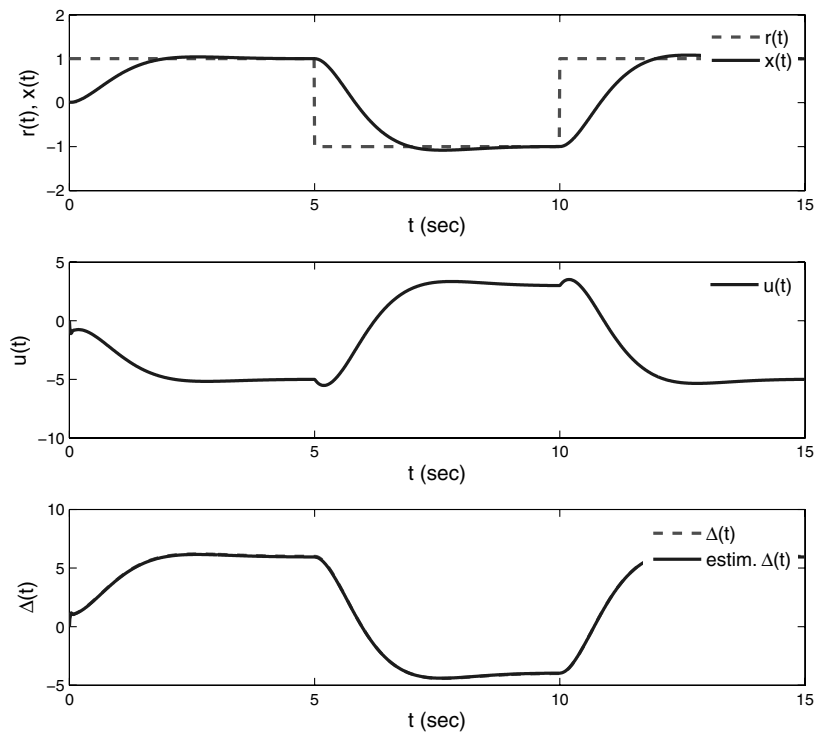

Fig. 12   Responses with DF-MRAC for an uncertainty with constant ideal weights.
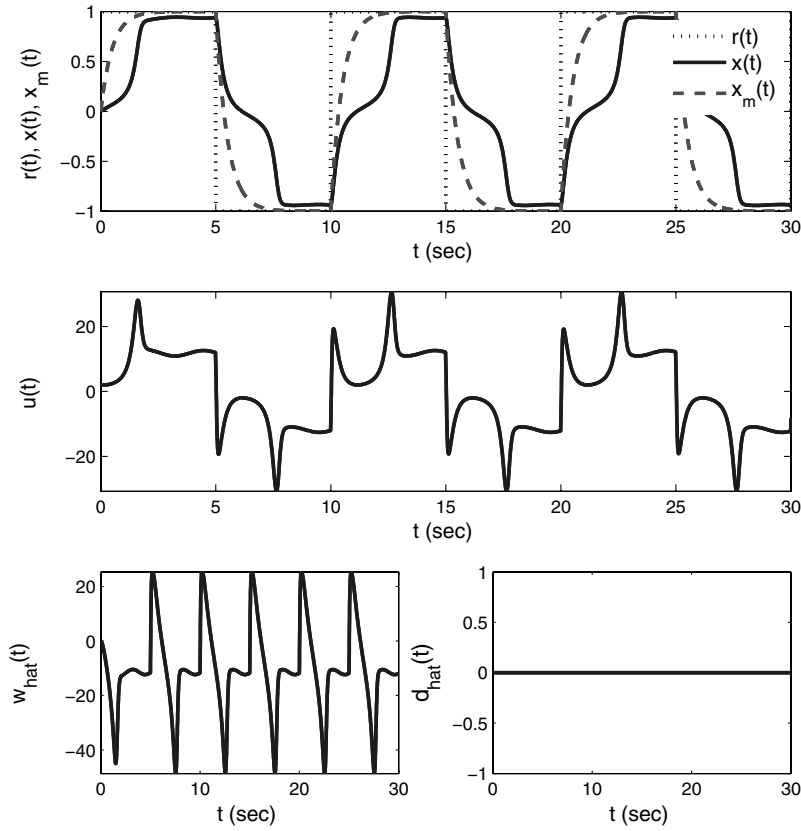
Fig. 13   Responses with DF-MRAC for input uncertainty case when Eq. (19) is employed.
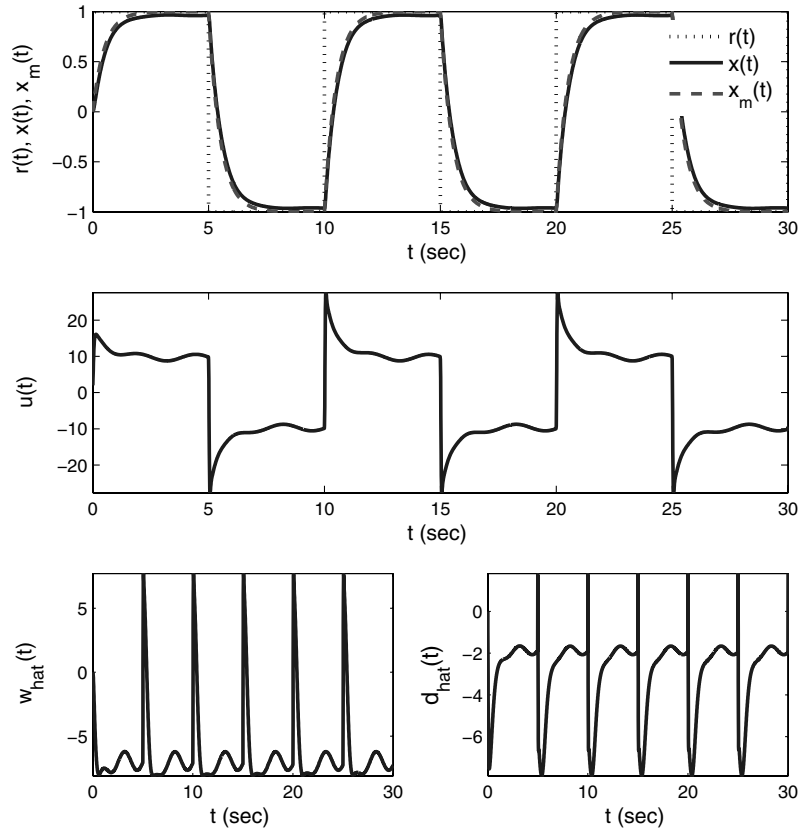

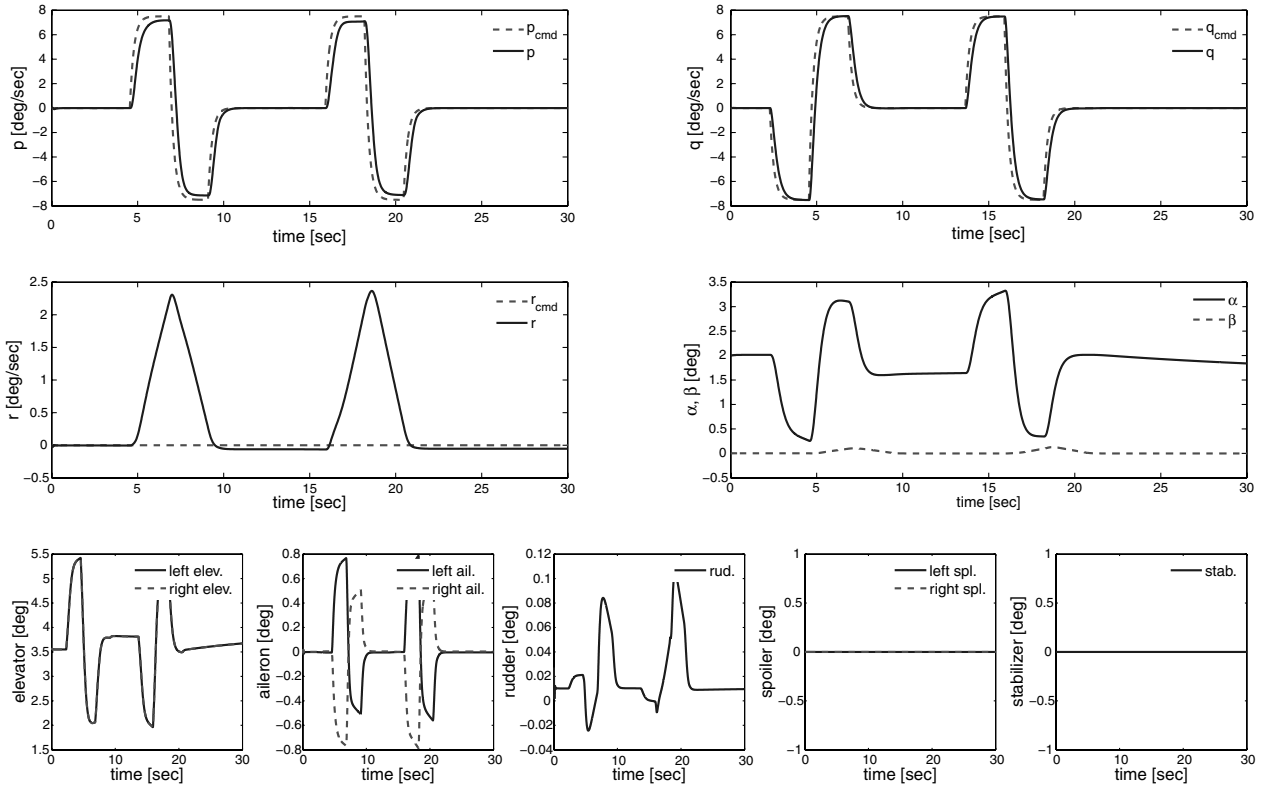Fig. 14   Responses with DF-MRAC for input uncertainty case when Eq. (58) is employed.

Fig. 15 GTM nominal control response for nominal operating conditions.

persistent. Figure 12 shows that the DF-MRAC law works well, even during the first transient phase.

C. Input Uncertainty Case

Consider the scalar dynamics

$$\dot{x}(t) = L_p x(t) + L_\delta \Lambda[u(t) + \Delta(x(t))] \tag{66}$$

where $\Lambda = 0.1$ represents an input uncertainty. Let $w(t)$ be a sinusoidal time-varying ideal weight with an amplitude of 1 and a frequency of 2.5 s. Furthermore, consider the same adaptive control design in Sec. VI.A for Eq. (19), and choose $\varphi_{d_i} = 0.5$, $\gamma_{d_i} = 50$, and $\tau_{d_i} = 0.01$ ($i = 1$) for Eq. (58). Figure 13 shows the result when (19) is employed and Fig. 14 shows the result when Eq. (58) is employed. There is a dramatic improvement when Eq. (58) is employed.
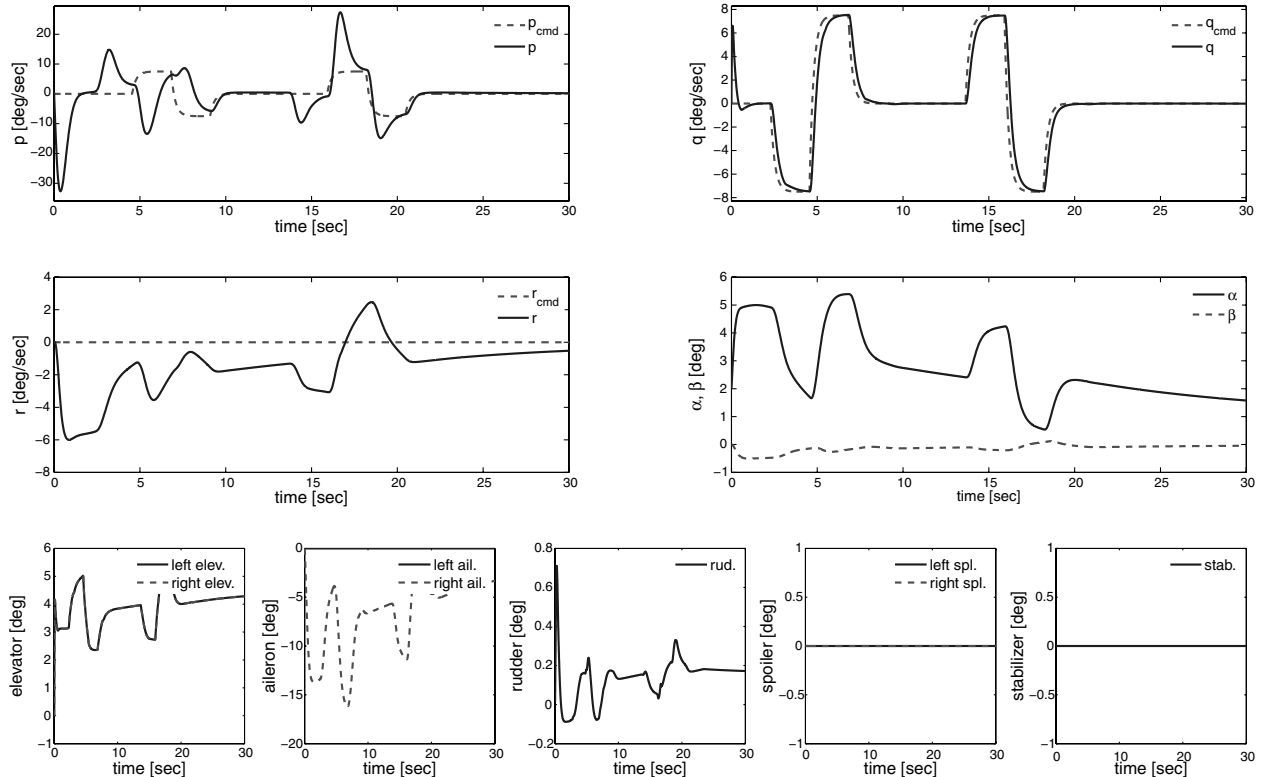

Fig. 16 GTM nominal control response for the missing-left-wingtip case.
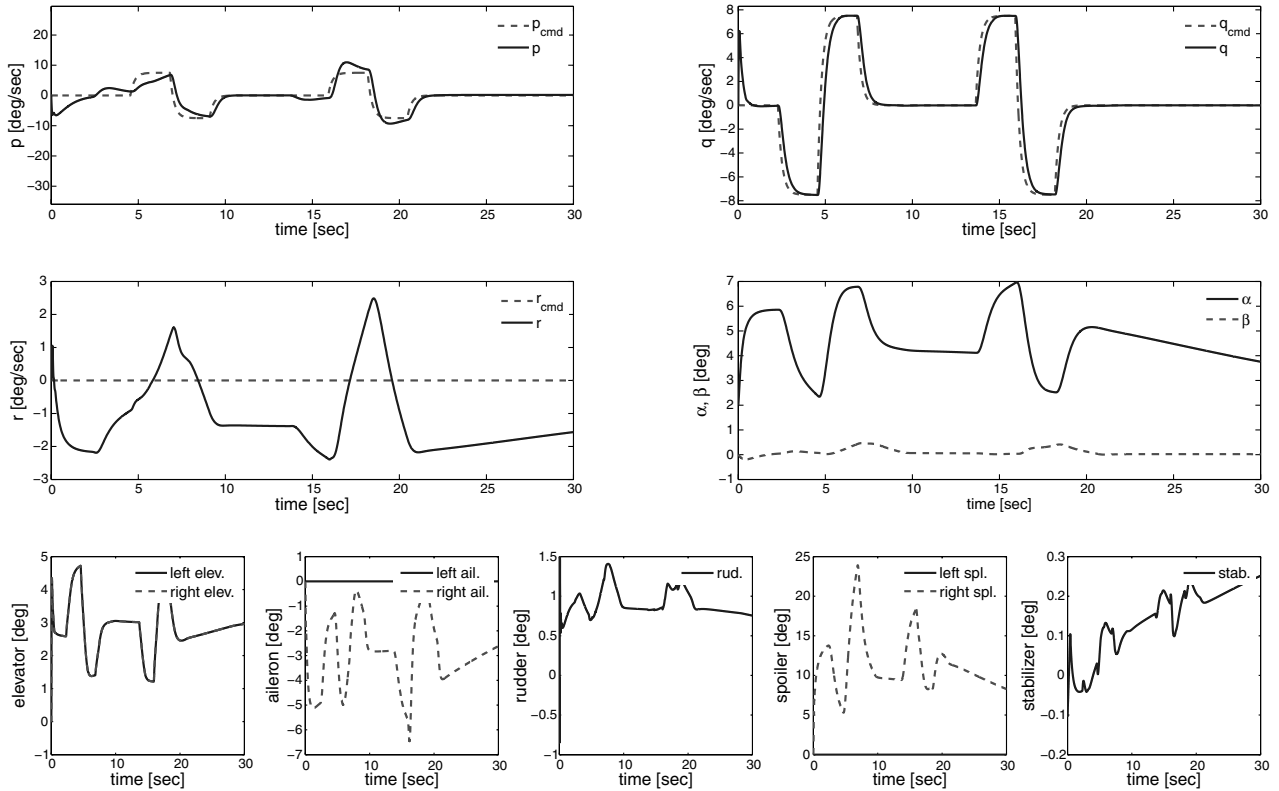
Fig. 17   GTM DF-MRAC response for the missing-left-wingtip case.

VII.   Generic Transport Model Example

This section presents a DF-MRAC design for the NASA GTM and evaluates this design for several damage scenarios. GTM is a high-fidelity scaled transport aircraft model developed by NASA Langley Research Center [15]. A linearized model for GTM at an angle of attack of 2 deg and 10^4 ft altitude is obtained in the form of Eq. (1).

The primary sources of uncertainty are any one of a set of possible damage conditions that are included as a part of the modeling in GTM. Flight-test results are reported in [18].

A nominal controller is first designed for the linearized model using a robust servomechanism linear quadratic regulator approach that incorporates integral control [33], with the objective of tracking
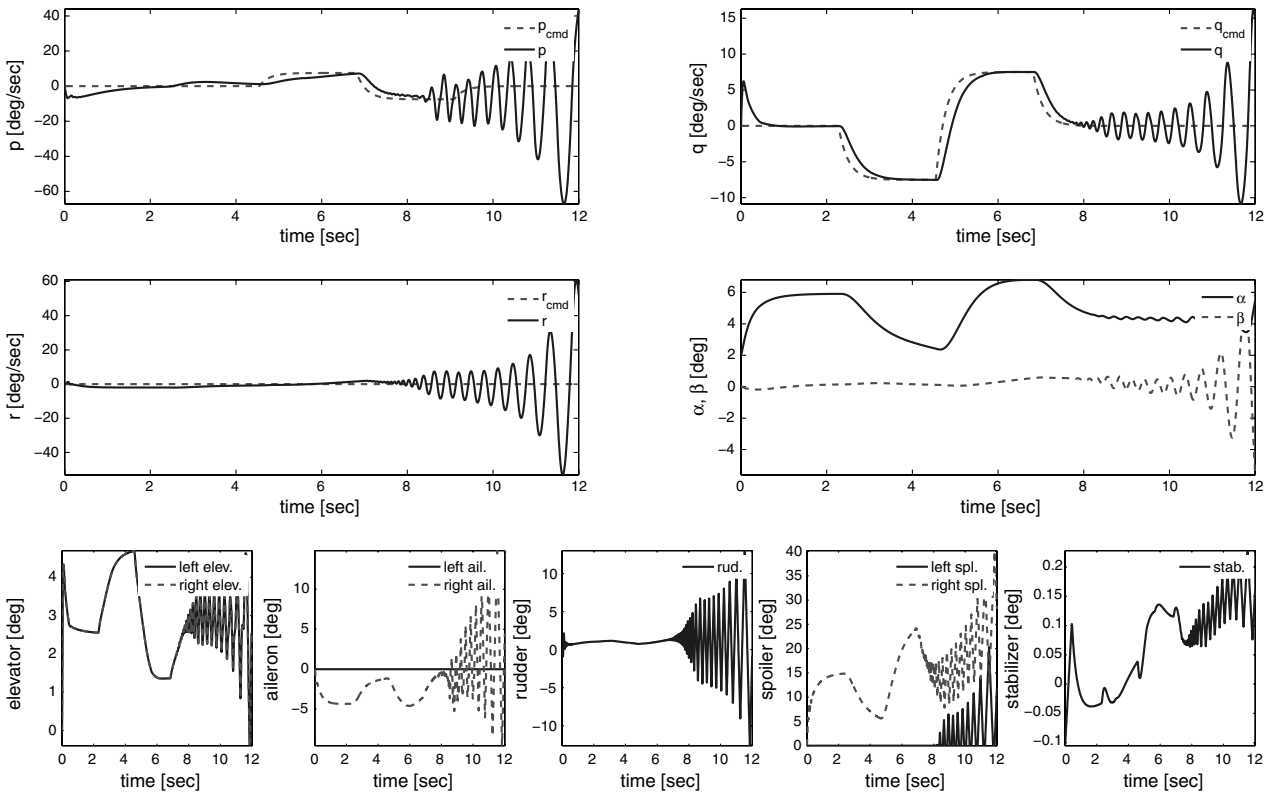

Fig. 18   GTM DF-MRAC response for the missing-left-wingtip case with 0.01 s of time delay in the rudder channel.
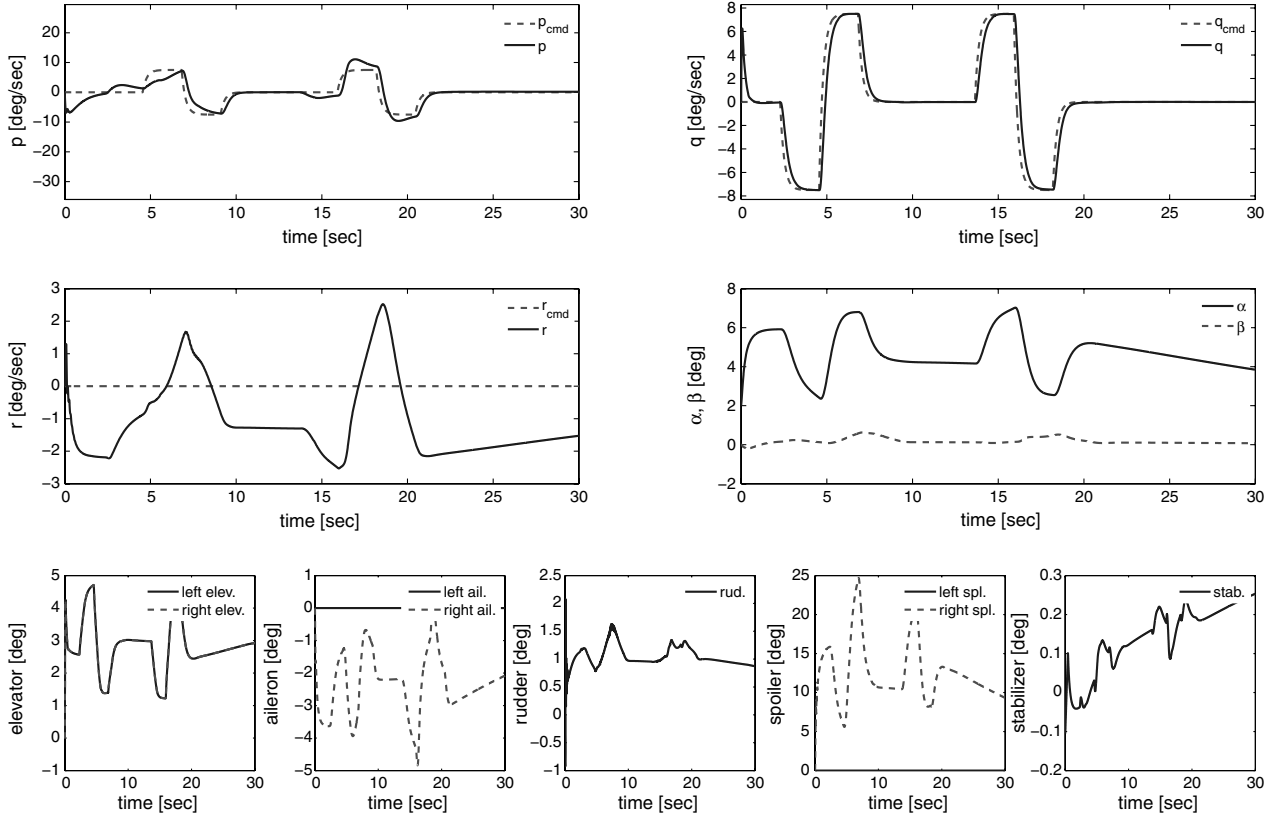
Fig. 19 GTM DF-MRAC response with ALR modification term for the missing-left-wingtip case with 0.01 s of time delay in the rudder channel.

roll rate, pitch rate, and yaw rate commands. Including the integral states, the linearized GTM model is ninth-order with the state vector:

$$x(t)$$
$$= [u(t)\ v(t)\ w(t)\ p(t)\ q(t)\ r(t)\ \phi(t)\ \theta(t)\ q_{int}(t)\ p_{int}(t)\ w_{int}(t)]^T$$

where $u(t)$, $v(t)$, and $w(t)$ are velocity components; $p(t)$, $q(t)$, $r(t)$ are body angular rates about the roll, pitch, and yaw body axes; $\phi(t)$ and $\theta(t)$ are roll and pitch attitude; and $q_{int}(t)$, $p_{int}(t)$, and $w_{int}(t)$ are the integrator states. In this simulation study, tracking of roll and pitch rate commands are considered, and yaw rate command is set to zero. Roll and pitch attitude are not used in the design. Figure 15 shows the performance of the nominal controller under normal operating conditions.

Since this design has redundant actuation and damage conditions that may include loss of one or more actuator channels, it is necessary to generalize the form in Eq. (55) to

$$\dot{x}(t) = Ax(t) + B\Lambda(G(I - \hat{D}(t))u_n(t) - \hat{W}^T(t)\beta(x(t)))$$
$$+ \Delta(x(t)) \tag{67}$$

where $G \in \mathbb{R}^{M \times 3}$ is a control allocation matrix (where $M > 3$ denotes the number of independent control channels); $u_n \in \mathbb{R}^3$ is the *effective* nominal control for the roll, pitch, and yaw axes, respectively; and $u_a \in \mathbb{R}^M$ is the adaptive control. Note that although the nominal control law must operate through the control allocation matrix, a portion of the adaptive controller has direct access to each independent channel of actuation. The quantity $\Lambda \in \mathbb{R}^{M \times M}$ is nominally an identity matrix, loss of actuation is represented by setting one or more of its diagonal elements to zero, and $\Delta: \mathbb{R}^n \to \mathbb{R}^n$ is the state-dependent uncertainty, which primarily enters the p, q, and r state equations. It is used to represent uncertainty in the stability derivatives. In general, the portion of the uncertainty that remains matched under actuator failure corresponds to the projection of Δ onto the column space of $B\Lambda$. For aircraft flight control applications the assumption that Δ remains matched under actuator failure amounts to assuming that $B\Lambda$ and Δ primarily influence the moments acting

on the aircraft and that control of moments in all three axes is maintained under actuator failure. For this study only the spoilers are independently controlled, so there is a total of $M = 6$ independent control channels: elevator, aileron, rudder, left and right spoilers, and stabilizer. Servo dynamics and position and rate limits are included in the model. The stabilizer servo has a relatively low bandwidth and low value for its rate limit and is useful primarily for maintaining trim in the pitch axis. The nominal control design is performed using BG in place of B in Eq. (1) when doing the design.

For the adaptive design, neural network sigmoidal-type functions [34] are used in the form

$$\beta(x(t)) = [1, \beta_1(x(t)), \beta_2(x(t)), \dots, \beta_6(x(t))]^T$$

where $\beta_i(x(t)) = (1 - e^{-x_i(t)})/(1 + e^{-x_i(t)})$ $(i = 1, \dots, 6)$, and P in Eq. (9) is found by using

$$Q = \text{diag}[\,10^{-4}\quad 10^{-2}\quad 10^{-3}\quad 60\quad 30\quad 15\quad 10\quad 10\quad 10\,]$$

Taking into consideration Theorem 4 and Table 1, weight update laws with ALR modification are

$$\hat{W}(t) = \Omega_1 \hat{W}(t - \tau) + [\beta(x(t))e^T PB$$
$$- \kappa_m \beta_x(x(t))\beta_x^T(x(t))\hat{W}(t)]\Gamma_{\kappa_2}, \qquad \Gamma_{\kappa_2} > 0 \tag{68}$$

$$\hat{d}_i(t) = \varphi_{d_i} \hat{d}_i(t - \tau_{d_i}) + \gamma_{d_i}[u_{n_i}(t)e^T(t)PB_i - \gamma_{m_i}\hat{d}_i(t)],$$
$$i = 1, 2, 3 \tag{69}$$

We chose $\Omega_1 = 0.1$, $\Gamma_{\kappa_2} = \text{diag}[\,10\quad 1\quad 10\quad 10\quad 50\quad 10\,]$, and $\tau = 0.5$ for Eq. (68) and $\varphi_{d_i} = 0.1$, $\gamma_{d_i} = 1.0$, and $\tau_{d_i} = 0.5$ $(i = 1, 2, 3)$ for Eq. (69). The ALR modification gains in Eqs. (68) and (69) were chosen as $\kappa_m = 2.5$ and $\gamma_{m_i} = 2.5$, respectively.

Remark 13. The ALR modification term in Eq. (69) has the same form as σ modification due to the fact that the basis functions are linear in $u_{n_i}(t)$.
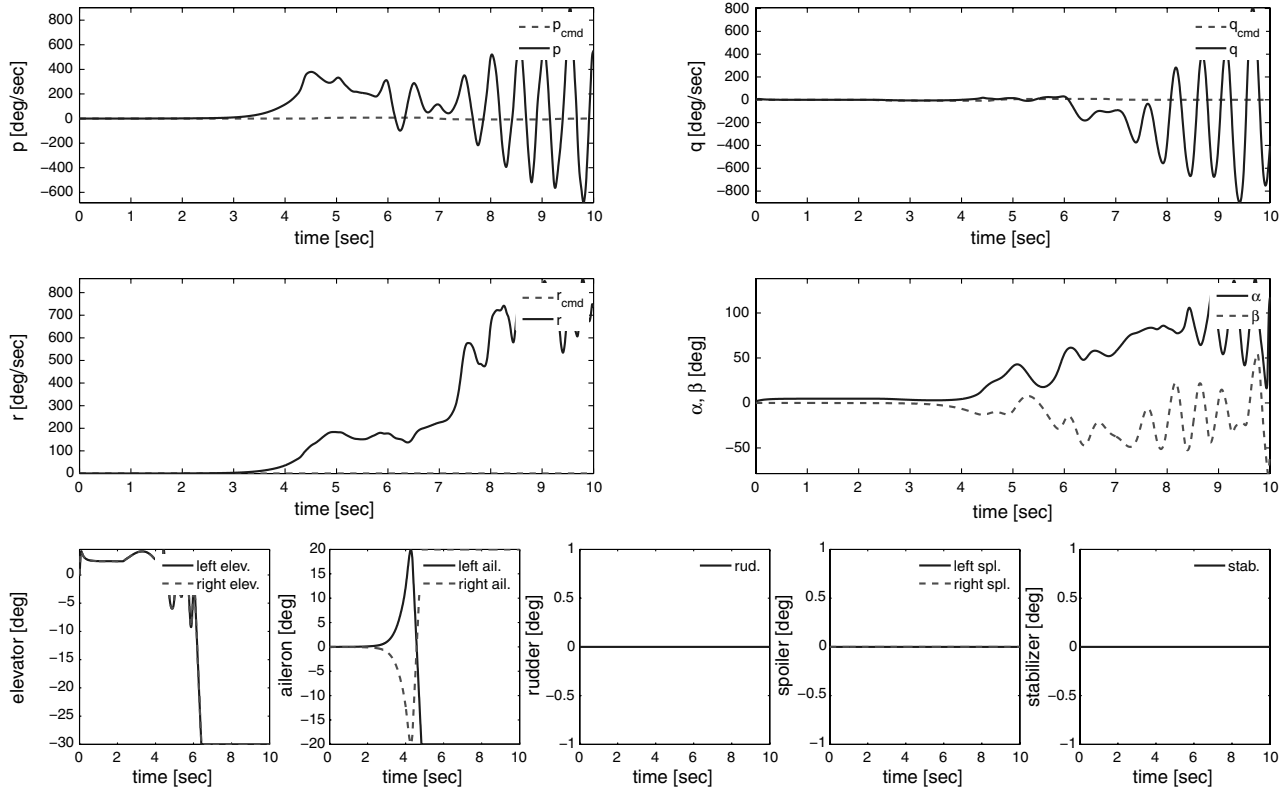
Fig. 20 GTM nominal control response for the missing-vertical-tail case.

Remark 14. The form in Eq. (68) is different from the one given in the fourth row of Table 1 in that Eq. (68) uses a matrix adaptation gain Γ_{κ_2}. In this case, the closed-loop system given by Eqs. (23) and (24) is still UUB by considering the Lyapunov–Krasovskii functional:

$$\mathcal{V}(e(t), \tilde{W}_t) = e^T(t)Pe(t) + \text{tr}\left[\int_{t-\tau}^{t} \tilde{W}^T(s)\tilde{W}(s)\Gamma\rho \, ds\right] \quad (70)$$

where $\Gamma_\rho > 0$. We choose a matrix adaptation gain rather than a scalar adaptation gain to add flexibility to the adaptive control design.
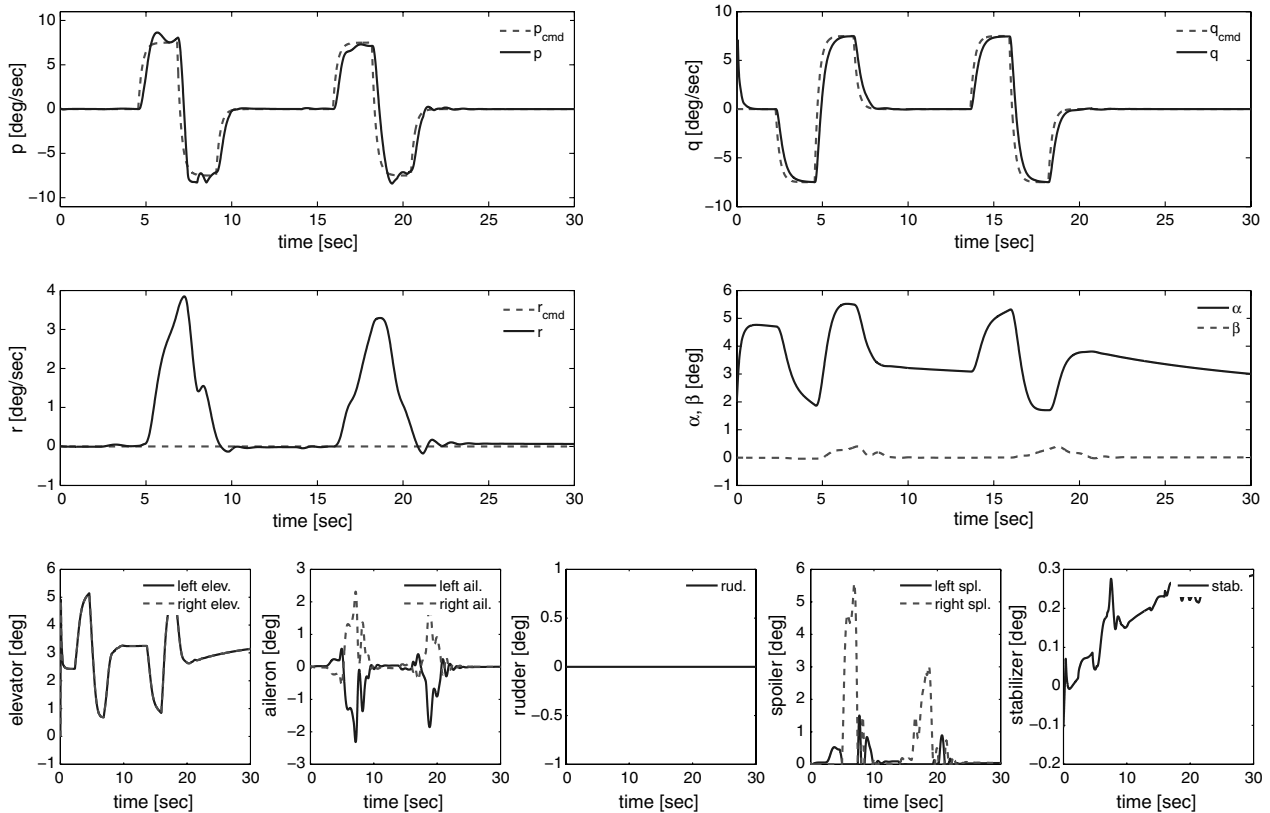

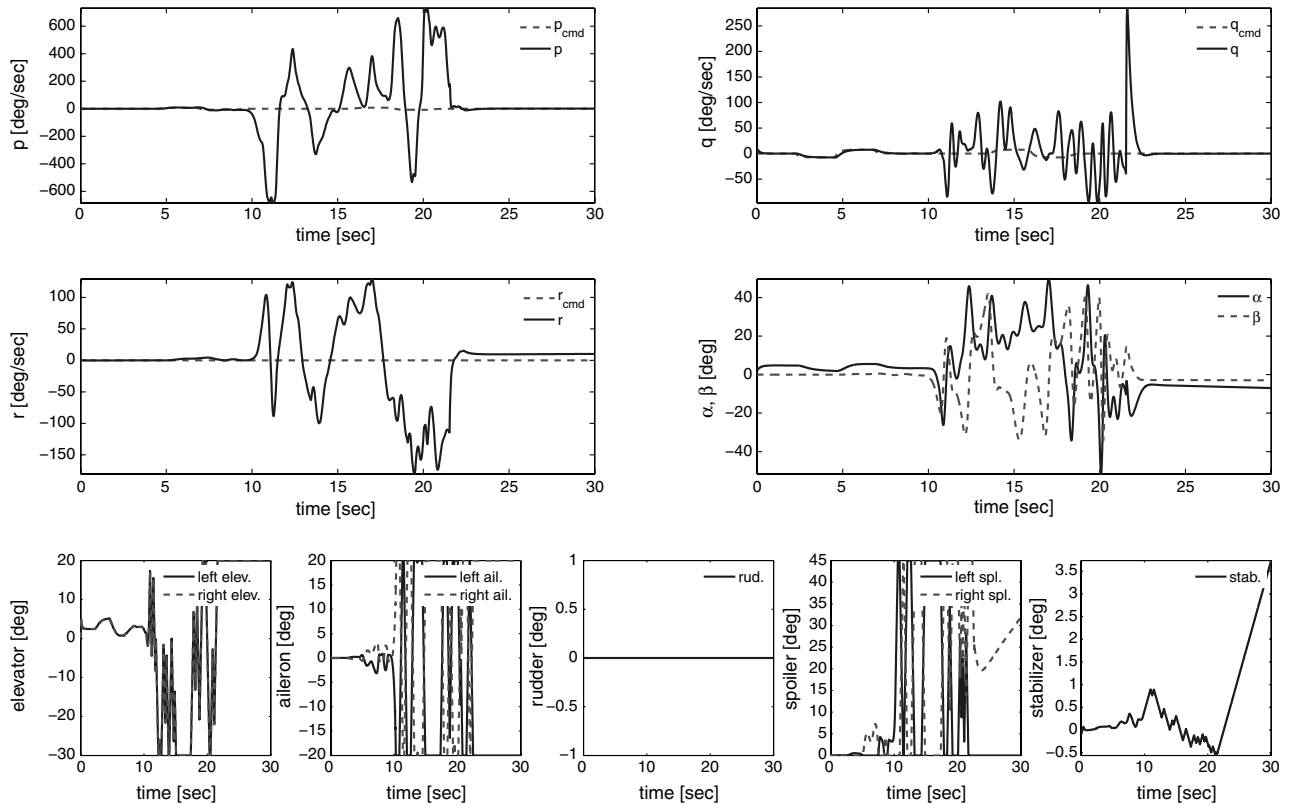**Fig. 21 GTM DF-MRAC response for the missing-vertical-tail case.**

Fig. 22   GTM DF-MRAC response for the missing-vertical-tail case with 0.07 s of time delay in the right aileron channel.

A.   Missing Left Wingtip

In the missing-left-wingtip damage scenario, there is 25% loss of outboard left wingtip and the left aileron is missing; therefore, available roll control effectiveness is reduced [15]. Figure 16 shows

the degree of degraded performance when the nominal controller is evaluated for this damage case. Figure 17 shows the improvement in response obtained when DF-MRAC is employed. In Fig. 18, 0.01 s of time delay is added to the rudder channel. In this case, the
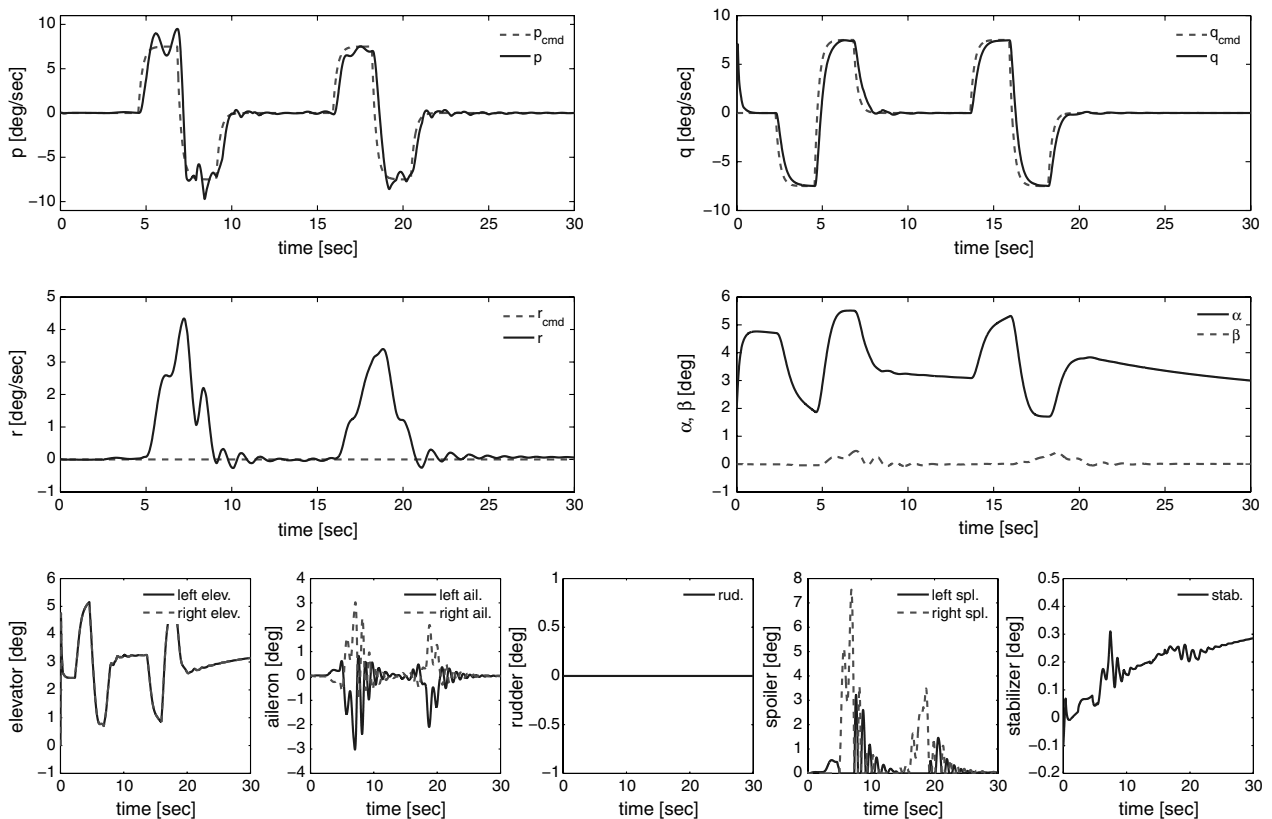

Fig. 23   GTM DF-MRAC response with ALR modification term for the missing-vertical-tail case with 0.07 s of time delay in the right aileron channel.

performance using DF-MRAC alone is not satisfactory. Figure 19 shows that the addition of ALR modification significantly improves the robustness of the adaptive controller to time delay.

B. Missing Vertical Tail

In the missing-vertical-tail damage scenario, the entire vertical tail is missing; therefore, there is a loss in directional stability and a complete loss in rudder control effectiveness [15]. Figure 20 shows that the nominal controller response for this damage case is unstable. Figure 21 shows that the DF-MRAC controller provides upset recovery and satisfactory tracking performance for this damage scenario. Figure 22 shows a result for the same damage case in Fig. 21 with 0.07 s of time delay in the right aileron channel. The performance of the DF-MRAC controller is not satisfactory with this amount of time delay. Figure 23 shows that ALR modification improves the time-delay margin of the DF-MRAC controller design for this failure case.

VIII.   Conclusions

This paper presents a derivative-free model reference adaptive control law for uncertain systems. The key feature is that the stability analysis is performed under the assumption that the ideal weights are bounded, but otherwise arbitrarily time-varying. The approach is particularly useful for situations in which the nature of the system uncertainty cannot be adequately represented by a set of basis functions with unknown constant weights. The system error signals, including the state tracking error and the weight update error, are proven to be uniformly ultimately bounded using a Lyapunov–Krasovskii functional without the introduction of modification terms. It is shown that the errors approach the ultimate bound exponentially in time. An analysis is provided that shows how the design parameters in the adaptive law can influence the ultimate bound and the exponential rate of convergence to the bound. The examples illustrate the superior performance of derivative-free adaptation in comparison with a conventional adaptive law, when it is evaluated for a variety of cases in which there is a sudden or rapidly varying set of dynamics. It is also shown how the derivative-free adaptive law can be modified using many of the existing modification terms. The simulation results show that dramatic improvement in robustness to time delays can be achieved by modifying the derivative-free adaptation law using a method of adaptive loop recovery.

Acknowledgment

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